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MORE ON SIGNED GRAPHS WITH AT MOST THREE EIGENVALUES

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Abstract

We consider signed graphs with just 2 or 3 distinct eigenvalues, in particular (i) those with at least one simple eigenvalue, and (ii) those with vertex-deleted subgraphs which themselves have at most 3 distinct eigenvalues. We also construct new examples using weighing matrices and symmetric 3-class association schemes.

Keywords: adjacency matrix, simple eigenvalue, strongly regular signed graph, vertex-deleted subgraph, weighing matrix, association scheme.

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1. Introduction

A *signed graph* $\dot{G}$ is defined to be a pair (G, σ), in which $G = (V, E)$ is an unsigned graph, called the *underlying graph*, and $\sigma\colon E \longrightarrow \{1, -1\}$ is the *sign function*, also known as the *signature*. The *order* of a signed graph, denoted by n, is the number of its vertices. The edge set of $\dot{G}$ consists of the subsets of positive and negative edges. We interpret an unsigned graph as a signed graph in which all edges are positive.

The *adjacency matrix* $A_{\dot{G}}$ of $\dot{G}$ is the $n \times n$ $(0, 1, -1)$-matrix which is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. By the *spectrum* of $\dot{G}$, we mean the spectrum of $A_{\dot{G}}$. An eigenvalue of $\dot{G}$ is called a *main* eigenvalue if the corresponding eigenspace is not orthogonal to the all-1 vector. Throughout the paper, by the statement '$\dot{G}$ has k eigenvalues' we mean that $\dot{G}$ has exactly k distinct eigenvalues.

In Section 2 we give some terminology and notation, and prove some auxiliary results. The problem of classifying graphs with a comparatively small number of eigenvalues has attracted a great deal of attention in the last 70 years; some recent results can be found in [4, 5, 7, 16, 17]. In [14] we considered regular and non-regular signed graphs with at most 3 eigenvalues; here we continue this research and pay more attention to non-regular signed graphs. In Section 3, we consider connected signed graphs with 3 eigenvalues at least one of which is simple. The number of simple eigenvalues governs our investigation in Section 4 of vertex-deleted subgraphs which themselves have 3 eigenvalues. In Section 5 we construct some signed graphs with 2 or 3 eigenvalues using weighing matrices or symmetric 3-class association schemes, and note the implications for vertex-deleted subgraphs.

2. Preliminaries

We write I, O, J, $\mathbf{0}$ and $\mathbf{j}$ for an identity matrix, an all-0 matrix, an all-1 matrix, an all-0 vector and an all-1 vector, respectively. Subscripts indicate size as necessary.

A signed graph $\dot{G}$ is said to be connected, complete, regular or bipartite if the same holds for its underlying graph. The degree of a vertex in $\dot{G}$ is the degree of the same vertex in G. The *net-degree* of a vertex i, denoted by $d_i^{\pm}$, is the difference between the numbers of positive and negative edges incident with i. A signed graph in which vertex net-degrees are equal is called *net-regular*. Similarly, a *net-biregular* signed graph is a signed graph which has 2 distinct net-degrees. It is known that $\dot{G}$ is net-regular if and only if $\mathbf{j}$ is an eigenvector of $\dot{G}$, and then $\mathbf{j}$ belongs to the eigenspace of the net-degree [20].

A signed graph is said to be *homogeneous* if all its edges have the same sign (in particular, if its edge set is empty). Otherwise, it is said to be *inhomogeneous*. The *negation* $-\dot{G}$ is obtained by reversing the sign of every edge of $\dot{G}$.

We say that signed graphs $\dot{G}$ and $\dot{H}$ are *isomorphic* if there is a permutation matrix P such that $A_{\dot{H}} = P^{-1}A_{\dot{G}}P$. In this case we write $\dot{G} \cong \dot{H}$. We say that $\dot{G}$ and $\dot{H}$ are *switching equivalent* if there is a vertex subset $S \subseteq V(\dot{G})$, such that $\dot{H}$ is obtained by reversing the sign of every edge with one vertex in S and the other in $V(\dot{G}) \setminus S$.

If the vertex labelling is transferred from the underlying graph common to $\dot{G}$ and $\dot{H}$, then $\dot{G}$ and $\dot{H}$ are switching equivalent if and only if there is a diagonal matrix D with ± 1 on the diagonal such that $A_{\dot{H}} = D^{-1}A_{\dot{G}}D$. Clearly, isomorphism and switching equivalence preserve the spectrum.

An *equitable partition* of a signed graph $\dot{G}$ is a partition of the vertex set $V(\dot{G})$ into non-empty cells $C_1, C_2, \ldots, C_s$, such that each cell induces a net-regular signed graph and for $1 \leq i < j \leq s$ the edges between C_i and C_j induce a net-biregular or net-regular signed graph, in which vertices from each of C_i, C_j are equal in net-degree.

We say that a signed graph $\dot{G}$ is *strongly regular* (for short, $\dot{G}$ is a *SRSG*) with parameters r, a, b, c if the entries of $A_{\dot{G}}^2$ satisfy

$$a_{ij}^{(2)} = \begin{cases} r & \text{if } i = j, \\ a & \text{if } i \overset{+}{\sim} j, \\ b & \text{if } i \overset{-}{\sim} j, \\ c & \text{if } i \not\sim j \text{ and } i \neq j. \end{cases}$$

Note that $a_{ij}^{(2)}$ is the difference between the numbers of positive and negative i-j walks of length 2 in $\dot{G}$. Accordingly, this definition generalizes the definition of strongly regular graphs. We mostly deal with SRSGs in Subsection 5.2.

In the forthcoming sections we frequently use the following result.

Proposition 1 [14]**.** *A connected signed graph $\dot{G}$ has exactly one positive eigenvalue if and only if $\dot{G}$ is switching equivalent to a non-trivial complete multipartite graph. If $\dot{G}$ has exactly one non-negative eigenvalue, then $\dot{G}$ is switching equivalent to a complete graph.*

We now transfer the following two results from the domain of unsigned graphs.

Proposition 2. *If A is a real symmetric matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that λ_1 is a simple eigenvalue, then*

$$\prod_{i=2}^{k}(A - \lambda_i I) = \left(\prod_{i=2}^{k}(\lambda_1 - \lambda_i)\right) \mathbf{x}\mathbf{x}^{\intercal},$$

where $\mathbf{x}$ *is a unit eigenvector associated with* λ_1. *For* $k = 3$, *there exists an eigenvector* $\mathbf{a}$ *for* λ_1, *such that* $(A - \lambda_2 I)(A - \lambda_3 I) = p\mathbf{a}\mathbf{a}^\intercal$, *where*

$$p = \begin{cases} 1 & \text{if } \lambda_1 \notin (\lambda_2, \lambda_3), \\ -1 & \text{if } \lambda_1 \in (\lambda_2, \lambda_3). \end{cases}$$

Proof. Considering the spectral decomposition of A, we see that there exists an orthogonal matrix X such that

$$\prod_{i=2}^{k}(A - \lambda_i I) = X \begin{pmatrix} \prod_{i=2}^{k}(\lambda_1 - \lambda_i) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} X^\intercal = \left(\prod_{i=2}^{k}(\lambda_1 - \lambda_i)\right)\mathbf{x}\mathbf{x}^\intercal,$$

where $\mathbf{x}$ is a unit eigenvector of $\prod_{i=2}^{k}(A - \lambda_i I)$ afforded by $\prod_{i=2}^{k}(\lambda_1 - \lambda_i)$. The result follows since $A\mathbf{x} = \lambda_1 \mathbf{x}$.

For $k = 3$, by taking $\mathbf{a} = \sqrt{p(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}\mathbf{x}$, we arrive at the desired result. ∎

The previous theorem is a slight extension of the result in which A is the adjacency matrix of an unsigned graph, $k = 3$ and λ_1 is the largest eigenvalue [4, 7, 16]. Our formulation is more general in order to embrace signed graphs, with the possibility that λ_1 is not the largest eigenvalue.

Proposition 3. *Let* $\dot{G}$ *be obtained from a signed graph* $\dot{H}$ *of order* n *by adding a new vertex whose neighbourhood in* $\dot{H}$ *is determined by the characteristic* $(0, 1, -1)$*-vector* $\mathbf{r}$. *The characteristic polynomial of* $\dot{G}$ *is given by*

$$P_{\dot{G}}(x) = P_{\dot{H}}(x)\left(x - \sum_{i=1}^{m}\frac{||Q_i\mathbf{r}||^2}{x - \mu_i}\right), \tag{1}$$

where $\mu_1, \mu_2, \ldots, \mu_m$ *are the distinct eigenvalues of* $\dot{H}$ *and* $Q_1, Q_2, \ldots, Q_m$ *are the matrices of the orthogonal projections of* $\mathbb{R}^n$ *onto the eigenspaces of* $\dot{H}$ *with respect to the canonical basis.*

Proof. Using the Schur matrix decomposition in conjunction with the known identity $\mathrm{adj}(xI - A_{\dot{H}}) = \det(xI - A_{\dot{H}})(xI - A_{\dot{H}})^{-1}$, we obtain

$$\begin{aligned} P_{\dot{G}}(x) = \det\begin{pmatrix} x & -\mathbf{r}^\intercal \\ -\mathbf{r} & xI - A_{\dot{H}} \end{pmatrix} &= xP_{\dot{H}}(x) - \mathbf{r}^\intercal \,\mathrm{adj}\left(xI - A_{\dot{H}}\right)\mathbf{r} \\ &= P_{\dot{H}}(x)\left(x - \mathbf{r}^\intercal(xI - A_{\dot{H}})^{-1}\mathbf{r}\right). \end{aligned}$$

Since $(xI - A_{\dot{H}})^{-1}$ has spectral decomposition $\sum_{i=1}^{m} \frac{1}{x-\mu_i} Q_i$, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x)\left(x - \mathbf{r}^{\intercal}\left(\sum_{i=1}^{m} \frac{1}{x-\mu_i} Q_i\right)\mathbf{r}\right).$$

Now (1) follows since $\mathbf{r}^{\intercal} Q_i \mathbf{r} = \mathbf{r}^{\intercal} Q_i Q_i \mathbf{r} = \mathbf{r}^{\intercal} Q_i^{\intercal} Q_i \mathbf{r} = (Q_i \mathbf{r})^{\intercal} Q_i \mathbf{r} = ||Q_i \mathbf{r}||^2$. ■

The 'unsigned' version of the previous theorem is well-known, see [6, Theorem 2.2.8]. The *cone* over a signed graph $\dot{G}$ is obtained by adding a vertex v along with positive edges between v and every vertex of $\dot{G}$. We denote this cone by $K_1 \nabla \dot{G}$. The following result is a direct consequence of the previous one.

Corollary 4. *The cone over $\dot{H}$ has the characteristic polynomial*

$$P_{K_1 \nabla \dot{H}}(x) = P_{\dot{H}}(x)\left(x - \sum_{i=1}^{m} \frac{n\beta_i^2}{x-\mu_i}\right),$$

where $\mu_1, \mu_2, \ldots, \mu_m$ are distinct eigenvalues of $\dot{H}$ and $\beta_1, \beta_2, \ldots, \beta_m$ are the corresponding main angles defined by $\beta_i = ||Q_i \mathbf{j}||/\sqrt{n}$.

3. Signed Graphs with 3 Eigenvalues, at Least One of which is Simple

In this section we give some characterizations of signed graphs described in the section title. We start with the following lemma.

Lemma 5. *If $\dot{G}$ is a connected signed graph with 3 eigenvalues such that at least 2 of them are simple, then $\dot{G}$ is switching equivalent to a complete bipartite graph.*

Proof. If every eigenvalue of $\dot{G}$ is simple, then $\dot{G}$ is switching equivalent to (the complete bipartite graph) $K_{1,2}$. Now suppose that λ is the unique non-simple eigenvalue. If $\lambda = 0$ then $\dot{G}$ has exactly one positive eigenvalue, hence is switching equivalent to a complete multipartite graph by Proposition 2.1. Moreover, since its spectrum has the form $[-\rho, 0^{n-2}, \rho]$, $\dot{G}$ is switching equivalent to a complete bipartite graph [6, p. 47].

If $\lambda \neq 0$ then $\dot{G}$ has a connected subgraph without λ as an eigenvalue, namely K_1. Since the eigenspace of λ has codimension 2, K_1 can be extended to a connected induced subgraph $\dot{H}$ of order 2 without λ as an eigenvalue (see [6, Theorem 5.1.6], which can be extended to the framework of signed graphs with slight modifications in the proof). Since $\dot{H} \cong \pm K_2$ we have $\lambda \notin \{1, -1\}$. Since also $\lambda \neq 0$, we know from [13, Theorem 3.3] that $\dot{G}$ has at most 4 vertices, and this case is resolved by inspection. ■

Now we consider signed graphs with 3 eigenvalues, exactly one of which is simple. Accordingly, we assume that a connected signed graph $\dot{G}$ has spectrum $[\rho, \mu^m, \lambda^l]$, with $m, l \geq 2$ and $\mu > \lambda$. By Proposition 2, there is a non-zero vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)^\intercal$ such that $A\mathbf{a} = \rho\mathbf{a}$ and

$$(A - \mu I)(A - \lambda I) = p\mathbf{a}\mathbf{a}^\intercal, \tag{2}$$

where $A = A_{\dot{G}}$, $p = -1$ if $\mu > \rho > \lambda$, and $p = 1$ otherwise. By equating the diagonal entries of both sides, we get

$$d_i = pa_i^2 - \mu\lambda, \tag{3}$$

where d_i is degree of the vertex i.

Lemma 6. *If $\dot{G}$ is a connected net-regular signed graph with spectrum $[\rho, \mu^m, \lambda^l]$, where ρ is its net-degree, then $\dot{G}$ is regular.*

Proof. Every eigenvector afforded by ρ is constant, which, by (3), means that $\dot{G}$ is regular. ■

We note in passing that, by [8], the signed graph $\dot{G}$ mentioned in the previous result is strongly regular. Moreover, we have the following result, which will be used in our last section.

Lemma 7. *A connected inhomogeneous non-complete regular signed graph $\dot{G}$ is net-regular with spectrum $[\rho, \mu^m, \lambda^l]$ (ρ being the net-degree) if and only if $\dot{G}$ is strongly regular and its parameters satisfy $a + b = 2c \neq 0$.*

Proof. Let $\dot{G}$ have spectrum as in the statement of the lemma. Since every eigenvector afforded by ρ is constant, by (2) we have

$$A^2 = (\mu + \lambda)A - \mu\lambda I + kJ, \quad \text{for some } k \neq 0.$$

Comparing the entries of the left and the right hand side, we conclude that $\dot{G}$ is strongly regular with $a + b = 2k$ and $c = k \neq 0$. The converse follows directly from [8, Theorem 4.2]. ■

Next we deal with the case in which an eigenvalue other than ρ is the only non-main eigenvalue.

Theorem 8. *If $\dot{G}$ is a connected signed graph with spectrum $[\rho, \mu^m, \lambda^l]$ ($m, l \geq 2$) such that only λ is non-main, then there is a non-zero constant α such that*

$$d_i = \alpha\big(d_i^{\pm} - \mu\big)^2 - \mu\lambda, \tag{4}$$

where d_i and $d_i^{\pm}$ are the degree and the net-degree of the vertex i, respectively.

In particular, if $\dot{G}$ is regular then it is net-biregular, the sum of the corresponding net-degrees is 2μ and $\dot{G}$ is switching equivalent to a net-regular signed graph. If $\dot{G}$ is biregular then it is switching equivalent to a net-biregular signed graph.

Proof. Observe first that $\dot{G}$ is not net-regular, since it has more than one main eigenvalue. We retain the notation introduced in Lemma 6. Since ρ and μ are main, we have

$$(A-\rho I)(A-\mu I)\mathbf{j}=\mathbf{0}, \tag{5}$$

as proved in [19] on the basis of the result for unsigned graphs which can be found in [15]. It follows that $A^2\mathbf{j}\in \operatorname{span}\langle \mathbf{d},\mathbf{j}\rangle$, where $A\mathbf{j}=\mathbf{d}=(d_1^{\pm},d_2^{\pm},\dots,d_n^{\pm})^{\intercal}$. Moreover, since $\mathbf{a}^{\intercal}\mathbf{j}\neq 0$, (2) shows that $\mathbf{a}\in \operatorname{span}\langle \mathbf{d},\mathbf{j}\rangle$. Hence, we may write $\mathbf{a}=r\mathbf{d}+s\mathbf{j}$, where $r\neq 0$ as $\dot{G}$ is not net-regular.

By (5), we have $A^2\mathbf{j}=(\rho+\mu)A\mathbf{j}-\rho\mu\mathbf{j}$, which together with (2), gives

$$(\rho-\lambda)\mathbf{d}-\mu(\rho-\lambda)\mathbf{j}=p\mathbf{a}(\mathbf{a}^{\intercal}\mathbf{j})=p(\mathbf{a}^{\intercal}\mathbf{j})(r\mathbf{d}+s\mathbf{j}).$$

By equating the coefficients of $\mathbf{d}$ and $\mathbf{j}$, we find that $s=-\mu r$, and so

$$\mathbf{a}=r(\mathbf{d}-\mu\mathbf{j}). \tag{6}$$

Using (3), we obtain $d_i=pr^2\big(d_i^{\pm}-\mu\big)^2-\mu\lambda$, and by setting $\alpha=pr^2$, we arrive at (4).

Now, if $\dot{G}$ is d-regular, then $d=\alpha(d_i^{\pm}-\mu)^2-\mu\lambda$ for every vertex i. Evidently, this equation has 2 solutions in $d_i^{\pm}$ and since $\dot{G}$ is not net-regular, both solutions appear as net-degrees; hence, $\dot{G}$ is net-biregular. The sum of the corresponding net-degrees follows from the previous equation for d. Lastly, from (3) we see that the coordinates of $\mathbf{a}$ are equal in absolute value. If D is the diagonal matrix of ± 1s with 1 in the ith position precisely when a_i is positive, then $D^{-1}AD$ is the adjacency matrix of a switching equivalent signed graph, say $\dot{H}$. Moreover, $D\mathbf{a}$ is a constant eigenvector associated with ρ in $\dot{H}$, which means that $\dot{H}$ is net-regular.

Finally, suppose that $\dot{G}$ is biregular with degrees d_1 and d_2, and assume that $\dot{G}$ is not net-biregular. Then, for at least one j ($j\in\{1,2\}$), there are vertices of degree d_j which differ in net-degree. By (6), the corresponding coordinates of $\mathbf{a}$ are different, while by (3), they are equal in absolute value. Using D formed exactly as before, we obtain a signed graph $\dot{H}$ for which we have $A_{\dot{H}}D\mathbf{a}=\rho D\mathbf{a}$ where $D\mathbf{a}$ has 2 different coordinates. By (6), this means that $\dot{H}$ has 2 net-degrees, and so it is net-biregular. ■

Of course, there is an analogous statement with μ in the role of the unique non-main eigenvalue. Here is a closer description of $\dot{G}$ being net-biregular.

Corollary 9. *If the signed graph $\dot{G}$ of Theorem 8 is net-biregular, then $\dot{G}$ is biregular and its net-degrees determine an equitable vertex bipartition.*

Proof. From (4) we see that $\dot{G}$ must be biregular. In addition, since $\dot{G}$ is non-regular, its vertices are equal in degree if and only if they are equal in net-degree. By (6), the eigenvector $\mathbf{a}$ of (2) has 2 different coordinates, say a_u and a_w, which correspond to different net-degrees and determine the vertex set partition $V = U \dot{\cup} W$. It remains to show that this partition is equitable. For $v \in V$, let $d_{vu}^{\pm}$ and $d_{vw}^{\pm}$ denote its net-degree in U and W, respectively. Then we also have $d_v^{\pm} = d_{vu}^{\pm} + d_{vw}^{\pm}$. If $v \in U$, since $\mathbf{a}$ is associated with ρ, we have $d_{vu}^{\pm}a_u + d_{vw}^{\pm}a_w = \rho a_u$, i.e., $d_{vu}^{\pm}a_u + \big(d_v^{\pm} - d_{vu}^{\pm}\big)a_w = \rho a_u$ and $\big(d_v^{\pm} - d_{vw}^{\pm}\big)a_u + d_{vw}^{\pm}a_w = \rho a_u$. The last two equalities lead to

$$d_{vu}^{\pm} = \frac{\rho a_u - d_v^{\pm}a_w}{a_u - a_w} \quad \text{and} \quad d_{vw}^{\pm} = a_u\frac{d_v^{\pm} - \rho}{a_u - a_w}.$$

In a very similar way, we obtain

$$d_{vu}^{\pm} = a_w\frac{d_v^{\pm} - \rho}{a_w - a_u} \quad \text{and} \quad d_{vw}^{\pm} = \frac{\rho a_w - d_v^{\pm}a_u}{a_w - a_u},$$

for $v \in W$. In other words, the net-degrees determine an equitable vertex bipartition. ■

It is not difficult to construct some examples. For instance, by making a switch with respect to 3 mutually adjacent vertices of the Paley graph with 9 vertices, we obtain a regular and net-biregular signed graph with spectrum $\big[4, 1^4, (-2)^4\big]$. Also, the cone over the complete bipartite signed graph $\dot{K}_{4,4}$, in which negative edges form a perfect matching, is biregular and net-biregular, while its spectrum is $\big[4, 2^3, (-2)^5\big]$.

4. Vertex-Deleted Subgraphs with 3 Eigenvalues

In this section we consider the question of whether a vertex-deleted subgraph of a connected signed graph $\dot{G}$ with 3 eigenvalues also has 3 eigenvalues. We distinguish 3 cases depending on the number of simple eigenvalues of $\dot{G}$. First, if all of them are simple then all vertex-deleted subgraphs have fewer than 3 eigenvalues; this case is trivial. If $\dot{G}$ has 2 simple eigenvalues, then $\dot{G}$ is switching equivalent to a complete bipartite graph, by Lemma 5. If so, then every vertex-deleted subgraph is also switching equivalent to a complete bipartite graph; such a subgraph has 3 eigenvalues unless $\dot{G}$ is the star $\dot{K}_{1,n-1}$ and the degree of the deleted vertex is $n-1$. The remaining case is more complicated and it is considered in the following two theorems.

Theorem 10. *Let $\dot{G}$ be a connected signed graph with spectrum $[\rho, \mu^m, \lambda^l]$, with $m, l \geq 2$ and $\mu > \lambda$. Let $\dot{H} = \dot{G} - v$ and let $\mathbf{r}$ be the characteristic $(0, 1, -1)$-vector that determines the neighbourhood of v in $\dot{H}$. If $\dot{H}$ has 3 eigenvalues, then $\mathbf{r}$ is an eigenvector associated with an eigenvalue of $\dot{H}$ distinct from μ and λ, and:*

(i) *for $\rho > \mu$, the spectrum of $\dot{H}$ is $[\mu^m, \rho + \lambda, \lambda^{l-1}]$ with $||\mathbf{r}||^2 = -\rho\lambda$;*

(ii) *for $\rho < \lambda$, the spectrum of $\dot{H}$ is $[\mu^{m-1}, \rho + \mu, \lambda^l]$ with $||\mathbf{r}||^2 = -\rho\mu$;*

(iii) *for $\rho \in (\lambda, \mu)$, the spectrum of $\dot{H}$ is $[\mu^{m-1}, \rho^2, \lambda^{l-1}]$ with $\rho = \mu + \lambda$, $||\mathbf{r}||^2 = -\mu\lambda$ or $[\mu^m, \rho + \lambda, \lambda^{l-1}]$ with $||\mathbf{r}||^2 = -\rho\lambda$, or $[\mu^{m-1}, \rho + \mu, \lambda^l]$ with $||\mathbf{r}||^2 = -\rho\mu$.*

Conversely, if $\mathbf{r}$ is an eigenvector of $\dot{H}$ associated with an eigenvalue distinct from μ and λ, then $\dot{H}$ has 3 eigenvalues when either ρ is an eigenvalue of multiplicity 2 in $\dot{H}$ or $\dot{H}$ does not have ρ as an eigenvalue.

Proof. Computing $\mathrm{tr}(A_{\dot{G}})$ and $\mathrm{tr}(A^2_{\dot{G}})$, we obtain

$$\rho + m\mu + l\lambda = 0, \tag{7}$$

$$\rho^2 + m\mu^2 + l\lambda^2 = 2e, \tag{8}$$

where e denotes the number of edges of $\dot{G}$. Let g be the number of edges in $\dot{G}$ but not in $\dot{H}$, so that $g = ||\mathbf{r}||^2$.

Suppose first that $\rho > \mu$. Observe that $\lambda < 0$, since $\dot{G}$ must have at least one negative eigenvalue (see Proposition 1), and then we also have $\mu \geq 0$. By eigenvalue interlacing, the eigenvalues of $\dot{H}$ are $\nu, \mu^{m-1}, \theta, \lambda^{l-1}$, where $\rho \geq \nu \geq \mu \geq \theta \geq \lambda$. Now, since $\dot{H}$ has 3 eigenvalues we have one of the following situations:

(a) $\nu = \rho$ and $\theta \in \{\mu, \lambda\}$,

(b) $\nu = \mu$ and $\theta \notin \{\mu, \lambda\}$,

(c) $\nu \notin \{\rho, \mu\}$ and $\theta = \mu$,

(d) $\nu \notin \{\rho, \mu\}$ and $\theta = \lambda$.

For (a), when $\theta = \mu$ we have $\rho + m\mu + (l-1)\lambda = 0$, which together with (7) leads to the contradiction $\lambda = 0$. When $\theta = \lambda$ in a similar way we get $\mu = 0$, but then $\dot{H}$ and $\dot{G}$ have the same number of edges (by (8)), a contradiction since $\dot{G}$ is connected.

For (b), we use the equalities (7) and (8) along with $\mathrm{tr}(A_{\dot{H}}) = 0$ and $\mathrm{tr}(A^2_{\dot{H}}) = 2(e - g)$ to obtain $\theta = \rho + \lambda$ and $g = -\rho\lambda$. To complete the proof of (i) it remains to prove that $\mathbf{r}$ is an eigenvector of $\dot{H}$ associated with $\rho + \lambda$.

Let Q_ξ denote the matrix of the orthogonal projection of $\mathbb{R}^{n-1}$ onto the eigenspace of an eigenvalue ξ of $\dot{H}$. By Proposition 3, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x)\left(x - \frac{||Q_\mu \mathbf{r}||^2}{x - \mu} - \frac{||Q_\theta \mathbf{r}||^2}{x - \theta} - \frac{||Q_\lambda \mathbf{r}||^2}{x - \lambda}\right).$$

Since the multiplicity of μ and λ in $\dot{G}$ is not less than the multiplicity of the same eigenvalue in $\dot{H}$, we have $Q_\mu\mathbf{r} = Q_\lambda\mathbf{r} = \mathbf{0}$, which means that $\mathbf{r}$ is orthogonal to the eigenspaces of μ and λ, equivalently $\mathbf{r}$ belongs to the eigenspace of $\theta = \rho + \lambda$.

For (c), as in the previous case, we obtain $\nu = \rho + \lambda$ and $g = -\rho\lambda$, along with the conclusion that $\mathbf{r}$ belongs the eigenspace of ν.

For (d), we find that $\nu = \rho + \mu$, which is impossible as $\mu \geq 0$ and $\nu \neq \rho$.

This completes the proof of (i), while (ii) follows analogously.

Now suppose that $\rho \in (\lambda, \mu)$. Here $\mu > 0, \lambda < 0$ and the possible eigenvalues of $\dot{H}$ are $\mu^{m-1}, \nu, \theta, \lambda^{l-1}$. The cases that arise are considered in the same way as before, and yield the results summarized in (iii).

Assume now that $\mathbf{r}$ belongs to the eigenspace of an eigenvalue of $\dot{H}$ distinct from μ and λ. If ρ is an eigenvalue of $\dot{H}$ with multiplicity 2, then (by eigenvalue interlacing) $\dot{H}$ has 3 eigenvalues, and so it remains to consider the case in which ρ does not belong to the spectrum of $\dot{H}$. In this case $\dot{H}$ has at most 4 eigenvalues. If λ, μ are the only eigenvalues of $\dot{H}$ then $\rho = 0$ and we obtain the contradiction $g = 0$. Now suppose that $\dot{H}$ has 4 eigenvalues, μ, λ, ν and θ. By Proposition 3, we have

$$P_{\dot{G}}(x) = P_{\dot{H}}(x)\Big(x - \frac{||Q_\mu\mathbf{r}||^2}{x-\mu} - \frac{||Q_\lambda\mathbf{r}||^2}{x-\lambda} - \frac{||Q_\nu\mathbf{r}||^2}{x-\nu} - \frac{||Q_\theta\mathbf{r}||^2}{x-\theta}\Big).$$

If $\mathbf{r}$ belongs to the eigenspace of (say) ν, then we obtain

$$(x-\nu)(x-\rho)(x-\mu)^m(x-\lambda)^l = \big(x(x-\nu) - ||Q_\nu\mathbf{r}||^2\big)P_{\dot{H}}(x). \tag{9}$$

Observe that the multiplicities of μ and λ in $\dot{H}$ are $m-1$ and $l-1$, respectively, which implies $(x-\mu)(x-\lambda) = \big(x(x-\nu) - ||Q_\nu\mathbf{r}||^2\big)$; but then since $(x-\rho)$ must appear on the right hand side of (9), we conclude that ρ is an eigenvalue of $\dot{H}$. This contradiction completes the proof. ∎

We note a consequence of Theorem 10 in the case that $\dot{G}$ is connected and switching equivalent to its underlying graph. Then ρ is the largest eigenvalue of $\dot{G}$, and so ρ is not an eigenvalue of $\dot{H}$. Hence, if $\mathbf{r}$ is an eigenvector of $\dot{H}$ associated with an eigenvalue other than μ or λ then $\dot{H}$ has 3 eigenvalues. Plenty of examples can be found among unsigned graphs; for instance $\dot{H}$ can be the Petersen graph, with $\dot{G}$ the cone over $\dot{H}$. We further consider cones by setting $\dot{G} \cong K_1\nabla\dot{H}$ in Theorem 10.

Corollary 11. *Suppose that $K_1\nabla\dot{H}$ has spectrum $[\rho, \mu^m, \lambda^l]$, where $m, l \geq 2$ and $\mu > \lambda$. If $\dot{H}$ is connected with 3 eigenvalues, then $\dot{H}$ is net-regular and either:*

(i) *$\dot{H}$ has spectrum $[\rho+\lambda, \mu^m, \lambda^{l-1}]$, with $\rho > \rho+\lambda > \mu > \lambda$ or*

(ii) *$\dot{H}$ has spectrum $[\mu^{m-1}, \lambda^l, \rho+\mu]$, with $\rho < \rho+\mu < \lambda < \mu$.*

Proof. By setting $\mathbf{r} = \mathbf{j}$ in Theorem 10 we see that $\mathbf{j}$ is an eigenvector of $\dot{H}$, and so $\dot{H}$ is net-regular. Let $A_{\dot{H}}\mathbf{j} = \nu\,\mathbf{j}$, $\dot{G} \cong K_1 \nabla \dot{H}$ and $n = 1 + l + m$. From Theorem 10 we see that $\nu \notin \{\mu, \lambda\}$ and there are five possible scenarios:

(a) $\rho > \mu$, $\nu = \rho + \lambda$, $n-1 = -\rho\lambda$ and $\dot{H}$ has spectrum $\left[\mu^m, \rho + \lambda, \lambda^{l-1}\right]$;

(b) $\rho < \lambda$, $\nu = \rho + \mu$, $n-1 = -\rho\mu$ and $\dot{H}$ has spectrum $\left[\mu^{m-1}, \rho + \mu, \lambda^{l}\right]$;

(c) $\lambda < \rho < \mu$, $\nu = \rho + \lambda$, $n-1 = -\rho\lambda$ and $\dot{H}$ has spectrum $\left[\mu^m, \rho + \lambda, \lambda^{l-1}\right]$;

(d) $\lambda < \rho < \mu$, $\nu = \rho + \mu$, $n-1 = -\rho\mu$ and $\dot{H}$ has spectrum $\left[\mu^{m-1}, \rho + \mu, \lambda^{l}\right]$;

(e) $\lambda < \rho < \mu$, $\nu = \rho$, $n-1 = -\lambda\mu$, $\rho = \mu + \lambda$ and $\dot{H}$ has spectrum $\left[\mu^{m-1}, \rho^2, \lambda^{l-1}\right]$.

Note first that if λ has multiplicity $l-1$ in $\dot{H}$ then $l > 2$, for otherwise $\dot{H}$ has 2 simple eigenvalues and, by Lemma 5, is switching equivalent to a complete bipartite graph, say $K_{r,s}$. Then $\mu = 0$ and $\{\nu, \lambda\} = \{-\sqrt{rs}, \sqrt{rs}\}$. Moreover, $r + s = n - 1 = -\rho\lambda = (\lambda - \nu)\lambda = 2rs$, whence $\dot{H} \cong K_2$, a contradiction. Similarly, if μ has multiplicity $m-1$ in $\dot{H}$, then $m > 2$. Now we may apply [14, Theorem 5.5] to $\dot{H}$, because $\dot{H}$ is net-regular, ν is a simple eigenvalue of $\dot{H}$ and each of λ, μ has multiplicity ≥ 2 in $\dot{H}$.

For (a), by [14, Theorem 5.5] either $\mu(\mu - \nu) = n - 1$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n-1$ and $\nu > \mu > 0 > \lambda$. In the former case, we have $\nu > \mu + \lambda$ and so $\lambda < \nu - \mu < \nu$, a contradiction. In the latter case, we have part (i) of this corollary.

For (b), we again get either $\mu(\mu - \nu) = n - 1$ and $\mu > 0 > \lambda > \nu$ or $\lambda(\lambda - \nu) = n-1$ and $\nu > \mu > 0 > \lambda$. In the former case, we have part (ii) of this corollary. In the latter case, μ is the largest eigenvalue of $\dot{G}$ because $\rho < \lambda$. Hence $\nu \leq \mu$, a contradiction.

For (c), we have $\lambda < 0$ and $\mu > 0$, because λ and μ are respectively the least and largest eigenvalues of $\dot{G}$. By [14, Theorem 5.5], either $\mu(\mu - \lambda) = n-1$ or $\lambda(\lambda - \mu) = n-1$. Since $\mu > \rho$, we have $-\lambda\mu > -\lambda\rho$, and so in the former case $\mu(\mu - \lambda) < -\lambda\mu$. Since $\mu > 0$, we have $\mu - \lambda < -\lambda$ and the contradiction $\mu < 0$. In the latter case, we have $\lambda(\lambda - \mu) = n-1 = -\rho\lambda$, whence $\rho = \mu - \lambda > \mu$, a contradiction.

For (d), we have $\lambda < 0$, $\mu > 0$ and either $\mu(\mu-\lambda) = n-1$ or $\lambda(\lambda-\mu) = n-1$. In the former case, $\mu(\mu - \lambda) = n - 1 = -\rho\mu$, whence $\rho = \lambda - \mu < \lambda$, a contradiction. In the latter case, $\rho > \lambda$ and so $\lambda(\lambda - \mu) = -\rho\mu < -\lambda\mu$, whence $\lambda - \mu > -\mu$ and the contradiction $\lambda > 0$.

For (e), we have $\nu = \lambda + \mu$, and so by [14, Theorem 5.5], $\dot{G}$ has just 2 eigenvalues, a contradiction. ■

It remains to consider signed graphs with 3 eigenvalues such that none of them is simple.

Theorem 12. *Let $\dot{G}$ be a connected signed graph with spectrum $\left[\rho^r, \mu^m, \lambda^l\right]$, with $r, m, l \geq 2$. Let $\dot{H} = \dot{G} - v$ and let $\mathbf{r}$ denote the characteristic $(0, 1, -1)$-vector that determines the neighbourhood of v in $\dot{H}$.*

If $\dot{H}$ has 3 eigenvalues then, to within a permutation of the eigenvalues, its spectrum is $\left[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}\right]$; moreover, $\rho = \mu + \lambda$, $||\mathbf{r}||^2 = -\mu\lambda$, and $\mathbf{r}$ belongs to the eigenspace of ρ in $\dot{H}$.

Conversely, if $\mathbf{r}$ belongs to the eigenspace of an eigenvalue of $\dot{H}$, then $\dot{H}$ has at most 4 eigenvalues, with 3 eigenvalues precisely when the eigenvalue associated with $\mathbf{r}$ also belongs to the spectrum of $\dot{G}$.

Proof. Assume that $\dot{H}$ has 3 eigenvalues and suppose first that the multiplicities of two of them (say μ and λ) are transferred from $\dot{G}$. Since $\mathrm{tr}(A_{\dot{G}}) = \mathrm{tr}(A_{\dot{H}})$ we have $\rho = 0$, and since $\mathrm{tr}(A^2_{\dot{G}}) = \mathrm{tr}(A^2_{\dot{H}})$ we see that $\dot{G}$ and $\dot{H}$ have the same number of edges, a contradiction. By eigenvalue interlacing, the remaining case is the one in which every eigenvalue changes its multiplicity: one increases and two decrease. If the multiplicity of ρ increases, then as before we have $\rho = \mu + \lambda$. Using Proposition 3 and following the proof of Theorem 10, we find that ρ is afforded by $\mathbf{r}$ in $\dot{H}$ and that $||\mathbf{r}||^2 = -\mu\lambda$.

Conversely, suppose that $\mathbf{r}$ belongs to the eigenspace of the eigenvalue ξ of $\dot{H}$. By Proposition 3, we have

$$(x - \xi)P_{\dot{G}}(x) = P_{\dot{H}}(x)\big(x(x - \xi) - ||Q_\xi \mathbf{r}||^2\big). \tag{10}$$

If $\dot{H}$ has 5 eigenvalues (more than this is impossible), those transferred from $\dot{G}$ along with (say) ν and θ, then at least one of the factors $(x-\nu)$ and $(x-\theta)$ occurs only on the right hand side of (10), which is impossible. Therefore, the number of eigenvalues of $\dot{H}$ is 3 or 4. If their number is 3, then we see immediately that ξ is an eigenvalue of $\dot{G}$. Conversely, if ξ is an eigenvalue of $\dot{G}$, and ν is the fourth eigenvalue of $\dot{H}$ (the one distinct from ρ, μ, λ), then as before $(x - \nu)$ occurs only on the right hand side of (10), which is impossible, and we are done. ∎

Example 13. To obtain an example for Theorem 12 it is convenient to start from $\dot{H}$ as a signed graph with spectrum $\left[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}\right]$, and set $\mathbf{r} = \mathbf{j}$. In this case, $\dot{H}$ is net-regular, and $\dot{G}$ is a cone over $\dot{H}$. Moreover, $\rho = \mu + \lambda$ and $\dot{H}$ has $-\mu\lambda$ vertices. By inspecting some known net-regular signed graphs with 3 eigenvalues, we arrive at a signed graph which can be found in [1] and satisfies all the numerical constraints. This is the signed graph obtained by reversing the sign of every edge belonging to a fixed Hamiltonian cycle of the Paley graph with 9 vertices. Its spectrum is $\left[3^2, 0^5, (-3)^2\right]$ (with $\rho = 0$). The corresponding cone has the spectrum $\left[3^3, 0^4, (-3)^3\right]$.

Motivated by the previous example, in which $\dot{G}$ is a cone over $\dot{H}$, we give a closer description of both signed graphs in this particular case. The first statement of the next theorem is a general one.

Theorem 14. *The following statements hold.*

(i) *Let $\dot{G}$ be a signed graph with ρ as an eigenvalue of multiplicity $r \geq 2$, let $\dot{H} = G - v$ and let $\mathbf{r}$ be the characteristic vector that determines the neighbourhood of v in $\dot{H}$. If the eigenspace of ρ in $\dot{H}$ has orthogonal basis $\mathbf{r}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ and v is vertex 1 in $\dot{G}$, then $\big(0, \mathbf{x}_1^\intercal\big)^\intercal, \big(0, \mathbf{x}_2^\intercal\big)^\intercal, \ldots, \big(0, \mathbf{x}_k^\intercal\big)^\intercal$ are linearly independent eigenvectors associated with ρ in $\dot{G}$.*

(ii) *In particular, if $\dot{G} \cong K_1 \nabla \dot{H}$ with spectrum $\big[\rho^r, \mu^m, \lambda^l\big](r, m, l \geq 2)$, if $\mathbf{j}$ belongs to the eigenspace of ρ in $\dot{H}$ and if $\dot{H}$ has spectrum $\big[\rho^{r+1}, \mu^{m-1}, \lambda^{l-1}\big]$, then ρ is the unique main eigenvalue of $\dot{H}$ and the unique non-main eigenvalue of $\dot{G}$.*

Proof. We have

$$A_{\dot{G}} \begin{pmatrix} 0 \\ \mathbf{x}_i \end{pmatrix} = \begin{pmatrix} \mathbf{r}^\intercal \mathbf{x}_i \\ \rho \mathbf{x}_i \end{pmatrix} = \rho \begin{pmatrix} 0 \\ \mathbf{x}_i \end{pmatrix},$$

for $1 \leq i \leq k$. Linear independence follows directly, and we have (i).

For (ii), first note that ρ is the unique main eigenvalue of $\dot{H}$ because $\mathbf{j}$ is an eigenvector of ρ in $\dot{H}$. Taking $k = r$ in (i) we obtain a basis for the eigenspace of ρ in $\dot{G}$ consisting of vectors orthogonal to $\mathbf{j}$. Thus ρ is non-main in $\dot{G}$. If $\dot{G}$ has another non-main eigenvalue, then the remaining one is main, but this means that the cone $\dot{G}$ is net-regular and hence the complete graph with just 2 eigenvalues, a contradiction. ∎

5. Constructions of Signed Graphs with at Most 3 Eigenvalues

Here we give some examples of signed graphs with 3 eigenvalues, along with applications of some results from Section 3. The first construction is based on weighing matrices, while the second one is based on symmetric 3-class association schemes.

5.1. Weighing matrices

Let $W = W(n, \alpha)$ be a weighing matrix of order n with weight α, i.e., an $n \times n$ $(0, 1, -1)$-matrix such that $W^\top W = \alpha I$. For $1 \leq m \leq n$, we call the submatrix of W indexed by rows $1, 2, \ldots, m$ and columns $1, 2, \ldots, n$ a *partial weighing matrix* with *weighing extension* W.

Theorem 15. *Let W_1' and W_2' be two partial weighing matrices of size $m \times n$ with weighing extensions W_1, W_2 of weight α. The following block matrix has spectrum $\big[-\sqrt{\alpha}^{n+m}, \sqrt{\alpha}^{n-m}, 2\sqrt{\alpha}^{m}\big]$:*

$$\mathcal{A}_m(W_1, W_2) = \begin{pmatrix} O_m & {W_1}' & {W_2}' \\ {W_1}'^{\intercal} & O_n & \frac{1}{\sqrt{\alpha}}W_1^{\intercal}W_2 \\ {W_2}'^{\intercal} & \frac{1}{\sqrt{\alpha}}W_2^{\intercal}W_1 & O_n \end{pmatrix}. \tag{11}$$

Proof. Let B be the matrix $({W_1}' \mid {W_2}')$, and let

$$C = \begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}}W_1^{\intercal}W_2 \\ \frac{1}{\sqrt{\alpha}}W_2^{\intercal}W_1 & O_n \end{pmatrix}.$$

By a straightforward computation we arrive at the following equality (see, for example, [10]):

$$(2\sqrt{\alpha}I_{2n} - C)^{-1} = \begin{pmatrix} \frac{2}{3\sqrt{\alpha}}I_n & \frac{1}{3\alpha\sqrt{\alpha}}W_1^{\intercal}W_2 \\ \frac{1}{3\alpha\sqrt{\alpha}}W_2^{\intercal}W_1 & \frac{2}{3\sqrt{\alpha}}I_n \end{pmatrix}.$$

Therefore $B(2\sqrt{\alpha}I_{2n} - C)^{-1}B^{\intercal} = 2\sqrt{\alpha}I_m$, and it follows that the vectors

$$\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B^{\intercal}\mathbf{x} \end{pmatrix},$$

for $\mathbf{x} \in \mathbb{R}^m$, lie in the eigenspace for the eigenvalue $2\sqrt{\alpha}$ of $\mathcal{A}_m(W_1, W_2)$. Thus $2\sqrt{\alpha}$ is an eigenvalue of $\mathcal{A}_m(W_1, W_2)$ with multiplicity at least m. A similar argument applied for $C = O_n$ shows that $-\sqrt{\alpha}$ is an eigenvalue with multiplicity at least $n + m$. In addition, by eigenvalue interlacing, $\sqrt{\alpha}$ is an eigenvalue of $\mathcal{A}_m(W_1, W_2)$ with multiplicity at least $n - m$ since the submatrix

$$\begin{pmatrix} O_n & \frac{1}{\sqrt{\alpha}}W_1^{\intercal}W_2 \\ \frac{1}{\sqrt{\alpha}}W_2^{\intercal}W_1 & O_n \end{pmatrix}$$

has the spectrum $[-\sqrt{\alpha}^n, \sqrt{\alpha}^n]$. ■

This theorem enables us to construct an infinite family of signed graphs with spectrum $\big[-\sqrt{\alpha}^{n+m}, \sqrt{\alpha}^{n-m}, 2\sqrt{\alpha}^m\big]$ for some appropriate α. We remark that the matrix (11) does not always correspond to a signed graph. For a signed graph, the inner product of any row of W_1 and any row of W_2 has to be 0 or $\pm\sqrt{\alpha}$, and in [11] one can find a method for constructing a family of weighing matrices of weight 4 with this property. The method can be used to construct signed graphs with spectrum $[-2^{n+m}, 2^{n-m}, 4^m]$, as in the following example.

Example 16. Let W_1 and W_2 be as follows:

$$W_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & 0 & -1 & 1 \\ 1 & -1 & -1 & 1 & 0 & 0 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & -1 & -1 \\ 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}.$$

Considering the first two rows of W_1 and W_2 as the matrices W_1' and W_2', we see from Theorem 15 that the signed graph $\dot{G}$ with adjacency matrix $\mathcal{A}_2(W_1, W_2)$ has spectrum $[-2^8, 2^4, 4^2]$. Since $\dot{G}$ has no vertices of degree 4, Theorem 4.3 shows that if a vertex-deleted subgraph of $\dot{G}$ has 3 eigenvalues, then it is obtained by deleting a vertex of degree 8. There are exactly two such vertices, labelled by 1 and 2 in (11), and the deletion of either leads to a subgraph with spectrum $\big[-2^7, 2^5, 4\big]$. This is because, in the notation of Theorem 4.3, $\mathbf{r}$ is an eigenvector of the vertex-deleted subgraph corresponding to the eigenvalue 2.

5.2. Symmetric 3-class association schemes

A symmetric 3-*class association scheme* $\mathcal{R}$ consists of a set X and a partition of $X \times X$ into 4 non-empty binary relations R_0, R_1, R_2, R_3 satisfying the following constraints:

- $R_0 = \{(x, x) \mid x \in X\}$;
- If $(x, y) \in R_i$, then $(y, x) \in R_i$ and if $(x, y) \in R_k$, then the number of $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant p_{ij}^k depending on i, j, k, but not on a particular choice of x, y.

For $0 \le i \le 3$, we define the $(0,1)$-matrix A_i with rows and columns indexed by the elements of X, and (x, y)-entry 1 if and only if $(x, y) \in R_i$. It follows that $A_0 = I$ and $A_iA_j = \Sigma_{k=0}^3 p_{ij}^k A_k$. For $i \in \{1, 2, 3\}$ let G_i be the graph with adjacency matrix A_i, and for distinct $i, j \in \{1, 2, 3\}$ let $\dot{G}_{i,j}$ be the signed graph with adjacency matrix $A_i - A_j$. The matrices A_i span a 4-dimensional commutative $\mathbb{R}$-algebra (called the Bose-Mesner algebra, cf. [3, Chapter 17]). It follows that the signed graphs $\dot{G}_{i,j}$ have at most 4 eigenvalues.

Theorem 17. *Let $\dot{G}_{i,j}$ be a signed graph arising from a 3-class association scheme. Then $\dot{G}_{i,j}$ is strongly regular with parameters*

$$r = p_{ii}^0 + p_{jj}^0, \; a = p_{ii}^i + p_{jj}^i - 2p_{ij}^i, \; b = -2p_{ij}^j + p_{ii}^j + p_{jj}^j, \; c = p_{ii}^k + p_{jj}^k - 2p_{ij}^k$$

where $\{i, j, k\} = \{1, 2, 3\}$. Moreover, $\dot{G}_{i,j}$ has 3 eigenvalues if and only if $a + b = 2c \neq 0$, and 2 eigenvalues if and only if $a + b = 2c = 0$.

Proof. The first assertion follows from [12, Theorem 2.2], and the second from Lemma 7. Lastly, $\dot{G}_{ij}$ is not a complete graph since $A_k \neq 0$, and so the third assertion follows from [18, Theorem 4.2]. ∎

We note that b is replaced by $-b$ in [12, Definition 1.4]. Now we are ready to provide some examples of signed graphs with at most 3 distinct eigenvalues.

Example 18. The 3-class *Johnson scheme* $J(n,3)$ $(n \geq 6)$, also known as the tetrahedral scheme, is defined on the 3-subsets of an n-set, with two subsets in the relation R_i if they intersect in $3-i$ elements. The following scheme provides the numbers p_{ij}^k relevant to $\dot{G}_{1,3}$; they are obtained by a simple computation.

p_{11}^1	p_{33}^1	p_{13}^1	p_{11}^3	p_{13}^3	p_{33}^3	p_{11}^2	p_{33}^2	p_{13}^2
$n-2$	$\binom{n-4}{3}$	0	0	$3(n-6)$	$\binom{n-6}{3}$	4	$\binom{n-5}{3}$	$n-5$

Therefore,

$$p_{11}^1 + p_{33}^1 - 2p_{13}^1 - 2p_{13}^3 + p_{11}^3 + p_{33}^3 = n-2+\binom{n-4}{3} - 6(n-6) + \binom{n-6}{3},$$

$$2(p_{11}^2 + p_{33}^2 - 2p_{13}^2) = 8 + 2\binom{n-5}{3} - 4n + 20.$$

Both of the above expressions are equal to $\frac{1}{3}(n-2)(n-7)(n-9)$. By Theorem 17, $\dot{G}_{1,3}$ has 2 eigenvalues when $n = 7$ or $n = 9$, and 3 eigenvalues otherwise.

Alternatively we can find the spectrum of $\dot{G}_{i,j}$ by using the following information from [2, 9]:

$$\mathrm{Spec}(A_1) = \left[3(n-3), (2n-9)^{n-1}, (n-7)^{\binom{n}{2}-n}, -3^{\binom{n}{3}-\binom{n}{2}}\right],$$

$$\mathrm{Spec}(A_3) = \left[\binom{n-3}{3}, (-n^2+9n)/2-10^{n-1}, (n-5)^{\binom{n}{2}-n}, -1^{\binom{n}{3}-\binom{n}{2}}\right].$$

Here the eigenvalues are ordered by common eigenvectors and so the eigenvalues of $A_1 - A_3$ are ρ, μ^{n-1} and $\lambda^{\binom{n}{3}-n}$, where ρ, μ, λ are not necessarily distinct, and

$$\rho = 3(n-3) - \binom{n-3}{3}, \quad \mu = \tfrac{1}{2}(n^2-5n+2), \quad \lambda = -2.$$

Note that G_1 is regular of degree $r_1 = 3(n-3)$, G_3 is regular of degree $r_2 = \binom{n-3}{3}$ and the graph underlying $\dot{G}_{1,3}$ is regular of degree $r_1 + r_2$.

We find that $\rho \geq \mu$ if and only if $n(n-2)(n-7) \leq 0$. When $n = 7$, $\dot{G}_{1,3}$ has spectrum $[8^7, -2^{28}]$ and by [14, Corollary 4.4] each vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues. Such a subgraph necessarily has spectrum $[6, 8^6, -2^{-27}]$. When $n = 6$ the spectrum of $\dot{G}_{1,3}$ is $[8, 4^5, -2^{14}]$ and each vertex of $\dot{G}_{1,3}$ has degree 10. Since $-\rho\lambda \neq 10$, Theorem 10(i) shows that no vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues.

Secondly, we find that $\rho \leq \lambda$ if and only if $(n-1)(n-2)(n-9) \geq 0$. When $n = 9$, the spectrum of $\dot{G}_{1,3}$ is $\left[19^8, -2^{76}\right]$ and by [14, Corollary 4.4] each vertex-deleted subgraph has spectrum $\left[17, 19^7, -2^{75}\right]$. When $n > 9$, no vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues for otherwise, in the notation of Theorem 10(ii), we have $\|\mathbf{r}\|^2 = -\rho\mu$; but $r_1 + r_2 = -\rho\mu$ if and only if $(n-1)(n-2)(n-9) = 0$, equivalently $n = 9$.

The remaining case is $n = 8$, when Theorem 10(iii) shows in similar fashion that a vertex-deleted subgraph of $\dot{G}_{1,3}$ does not have 3 distinct eigenvalues. In summary, a vertex-deleted subgraph of the signed graph $\dot{G}_{1,3}$ derived from $J(n,3)$ has 3 eigenvalues if and only if n is 7 or 9.

The definition of $J(n,3)$ may be extended to $J(5,3)$, but this scheme is degenerate in our context because then $A_3 = 0$. In fact, $J(5,3)$ is a 2-class association scheme, with $G_1 \cong L(K_5)$ and $G_2 \cong \overline{G}_1$ (the Petersen graph). Here $\dot{G}_{1,2}$ has spectrum $\left[-3^5, 3^5\right]$, while any vertex-deleted subgraph has spectrum $\left[-3^4, 0, 3^4\right]$.

Example 19. The 3-class *Hamming scheme* $H(3,q)$ is defined on the triples of q symbols (words of length 3 over an alphabet with q letters), where two triples are in the relation R_i if they differ in i coordinates ($i = 0, 1, 2, 3$). In this case we have the following.

p^1_{11}	p^1_{33}	p^1_{13}	p^3_{11}	p^3_{13}	p^3_{33}	p^2_{11}	p^2_{33}	p^2_{13}
$q-2$	$(q-2)(q-1)^2$	0	0	$3(q-2)$	$(q-2)^3$	2	$(q-2)^2(q-1)$	$q-1$

Therefore,

$$p^1_{11} + p^1_{33} - 2p^1_{13} - 2p^3_{13} + p^3_{11} + p^3_{33} = q - 2 + (q-2)(q-1)^2 - 6(q-2) + (q-2)^3,$$

$$2\left(p^2_{11} + p^2_{33} - 2p^2_{13}\right) = 4 + 2(q-2)^2(q-1) - 4q + 4.$$

Note that both of the above expressions are equal to $2q(q-2)(q-3)$. By Theorem 17, the signed graph $\dot{G}_{1,3}$ has at most 3 distinct eigenvalues. Moreover, $c = 0$ if and only if q is 2 or 3, and these are the cases in which $\dot{G}_{1,3}$ has only 2 distinct eigenvalues. Thus, by [14, Corollary 4.4] any vertex deleted subgraph of $\dot{G}_{1,3}$ has 3 distinct eigenvalues when $q \in \{2, 3\}$. On the other hand, by the proof of [8, Theorem 4.2] the eigenvalues of $\dot{G}_{1,3}$ other than its net-degree are the roots of the quadratic

$$x^2 + \frac{b-a}{2}x + \frac{a+b}{2} - r.$$

Now, by Theorem 17, we conclude that the eigenvalues of $\dot{G}_{1,3}$ are

$$\rho = 3(q-1) - (q-1)^3, \quad \mu = q^2 - 2, \quad \lambda = -2.$$

For $q > 3$, we find $\rho < \lambda$, and so by Theorem 10(ii), if a vertex-deleted subgraph of $\dot{G}_{1,3}$ has only 3 distinct eigenvalues then $\|\mathbf{r}\|^{\mathbf{2}} = -\rho\mu$. This equality holds if and only if

$$q(q-1)^2(q+2)(q-3)=0.$$

Accordingly, no vertex-deleted subgraph of $\dot{G}_{1,3}$ has 3 eigenvalues when $q>3$.

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