Rowlinson P (2014) On independent star sets in finite graphs, *Linear Algebra and Its Applications*, 442, pp. 82-91.

This is the peer reviewed version of this article

ON INDEPENDENT STAR SETS IN FINITE GRAPHS

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Abstract

Let G be a finite graph with μ as an eigenvalue of multiplicity k. A star set for μ is a set X of k vertices in G such that μ is not an eigenvalue of $G - X$. We investigate independent star sets of largest possible size in a variety of situations. We note connections with symmetric designs, codes, strongly regular graphs, and graphs with least eigenvalue -2.

AMS Classification: 05C50

Keywords: eigenvalue, error-correcting code, star set, strongly regular graph, symmetric design.

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1 Introduction

Let G be a finite simple graph of order n with μ as an eigenvalue of multiplicity k, and let $t = n - k$. Thus the corresponding eigenspace $\mathcal{E}(\mu)$ of a $(0,1)$-adjacency matrix of G has dimension k and codimension t. We call t the *co-multiplicity* of μ. A *star set* for μ in G is a subset X of the vertex-set $V(G)$ such that $|X| = k$ and the induced subgraph $G - X$ does not have μ as an eigenvalue. In this situation, $G - X$ is called a *star complement* for μ in G. We use the notation of [11], where the fundamental properties of star sets and star complements are established in Chapter 5. Recall that μ is said to be a *main* eigenvalue if $\mathcal{E}(\mu)$ is not orthogonal to the all-1 vector in $\mathbb{R}^n$, and that in an r-regular graph, all eigenvalues other than r are non-main.

It is well known that if $\mu \neq -1$ or 0 and $G \neq K_2$ or $2K_2$ then $|X| \leq \binom{t}{2}$; moreover, $|X| \leq \binom{t}{2} - 1$ when μ is not a main eigenvalue [3]. We shall soon see that if further X is an independent set then $|X| \leq t$, while $|X| \leq t - 1$ when μ is non-main. In Section 2 we investigate graphs with an independent star set X of size t, and note the role of symmetric 2-designs in an extremal configuration. In Section 3 we determine all the graphs that occur when $|X| = t$ and $\mu = -2$. In Section 4 we see how independent star sets of size $t - 1$ (for a non-main eigenvalue) can arise from strongly regular graphs. In Section 5 we show how smaller upper bounds for $|X|$ apply when a particular star complement is used to determine an error-correcting code.

The special case of an independent star set of size t for the non-main eigenvalue -1 features in [1, Proposition 4.2] (see also Proposition 2.1(ii) below). The authors of [2] investigate graphs in which every star set for every eigenvalue is independent; such graphs are called *galaxy graphs* [4]. In contrast, our approach here is to explore how a single independent star set can arise. Note that if S is a star set for μ in G and if U is a proper subset of S then (by interlacing) $S \setminus U$ is a star set for μ in $G - U$. We deduce that if the subset X of S is independent, then by removing from G the vertices of S outside X, we obtain a graph with X as an independent star set. Note also that we may confine our attention to maximal independent star sets; here we consider independent star sets of largest size in a variety of situations.

We shall require the following properties of star sets. For any $X \subseteq V(G)$, we write G_X for the subgraph of G induced by X. We take $V(G) = \{1, \ldots, n\}$, and write $u \sim v$ to mean that vertices u and v are adjacent. An all-1 vector is denoted by $\mathbf{j}$, its length determined by context.

Theorem 1.1 [11, Theorem 5.1.7] *Let X be a set of k vertices in G and suppose that G has adjacency matrix $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$, where A_X is the adjacency matrix of G_X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^\top(\mu I - C)^{-1}B. \tag{1}$$

In this situation, $\mathcal{E}(\mu)$ consists of the vectors $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1}B\mathbf{x} \end{pmatrix}$ $(\mathbf{x} \in \mathbb{R}^k)$.

Writing $H = G - X$, we see that the columns $\mathbf{b}_u$ $(u \in X)$ of B are the characteristic vectors of the H-neighbourhoods $\Delta_H(u) = \{v \in V(H) : u \sim v\}$ $(u \in X)$. Thus G is determined by μ, a star complement H for μ, and the H-neighbourhoods $\Delta_H(u)$ $(u \in X)$. Moreover, when $\mu \notin \{-1, 0\}$, these neighbourhoods are non-empty and distinct because Eq. (1) shows that

$$\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{b}_v = \begin{cases} \mu \text{ if } u = v \\ -1 \text{ if } u \sim v \\ 0 \ \text{ otherwise.} \end{cases} \tag{2}$$

From the description of $\mathcal{E}(\mu)$ in Theorem 1.1, we have the following result.

Proposition 1.2 [10, Proposition 0.3] *With the notation above, μ is a non-main eigenvalue if and only if*

$$\mathbf{b}_u^\top(\mu I - C)^{-1}\mathbf{j} = -1 \ \text{ for all } u \in X. \tag{3}$$

2 First observations

Let G be a graph with μ as a non-zero eigenvalue of co-multiplicity t, and suppose that X is an independent star set for μ in G. We use the notation of Theorem 1.1: from Eq.(1) we have $I = B^\top(\mu^2 I - \mu C)^{-1}B$, whence $|X| \leq \operatorname{rank}(\mu^2 I - \mu C) = t$.

We investigate the case $|X| = t$. In this situation, μ is an integer, for otherwise it has an algebraic conjugate which is a second eigenvalue of multiplicity t; but by [11, Theorem 3.3.5] each component of a graph with just two distinct eigenvalues is complete, giving a contradiction. We see also that the coclique on X is another star complement for μ, and we may apply Theorem 1.1 to the adjacency matrix

$$A^* = \begin{pmatrix} C & B \\ B^\top & O \end{pmatrix}$$

to obtain $BB^\top = \mu^2 I - \mu C$. Thus if $V(H) = \{t+1, \ldots, 2t\}$ and $B^\top = (\mathbf{q}_1 | \cdots | \mathbf{q}_t)$ then

$$\mathbf{q}_i^\top \mathbf{q}_j = \begin{cases} \mu^2 \text{ if } i = j \\ -\mu \text{ if } t+i \sim t+j \\ 0 \ \text{ otherwise.} \end{cases} \tag{4}$$

By interlacing, μ is either the smallest or the largest eigenvalue of G. In the latter case, G has at least t components, by [11, Corollary 1.3.8]. On the other hand, each vertex in X is adjacent to a vertex of H [11, Proposition 5.1.4], and so $G = tK_2$, $\mu = 1$.

If μ is a non-main eigenvalue of G then by Proposition 1.2, each $\mathbf{q}_i^\top \mathbf{j}$ is equal to $-\mu$. Since $\mathbf{q}_i^\top \mathbf{j} = \mathbf{q}_i^\top \mathbf{q}_i = \mu^2$, we have $\mu = -1$. Now -1 is the smallest eigenvalue of G, and so each component of G is complete. (Since $I + A$ is expressible in the form $M^\top M$, 'equality or adjacency' is a transitive relation on $V(G)$.) Thus again $G = tK_2$. Accordingly, we exclude the graph tK_2 from our considerations. So far, we have shown:

Proposition 2.1 *Let G be a graph with μ as a non-zero eigenvalue of co-multiplicity t, and suppose that X is an independent star set for μ in G. We have* (i) $|X| \leq t$, (ii) *if μ is non-main and $G \neq tK_2$ then $|X| \leq t-1$.*

We give an example to show that the first bound in Proposition 2.1 is sharp. Sharpness of the second bound will follow from results in Section 4.

Example 2.2 For a design $\mathcal{D}$, let $G(\mathcal{D})$ denote the graph obtained from the incidence graph of $\mathcal{D}$ by adding edges between all blocks. If $\mathcal{D}$ is the complement of the Fano plane then $G(\mathcal{D})$ is non-regular with spectrum $-2^{(7)}, 1^{(6)}, 8$; see [12, Chapter 2], where this graph is illustrated in Fig. 2.1.2. The points of $\mathcal{D}$ form an independent star set for -2, and the clique on the blocks of $\mathcal{D}$ is the corresponding star complement. $\square$

Theorem 2.3 *Let G be a graph with μ as a non-zero eigenvalue of co-multiplicity t, and suppose that G has an independent star set X for μ. If $|X| = t$ and $G \neq tK_2$ then μ is a negative integer, μ is a main eigenvalue, and $t \geq -\mu^3 + \mu + 1$; moreover, $t = -\mu^3 + \mu + 1$ if and only if $G = G(\mathcal{D})$ where $\mathcal{D}$ is a symmetric 2-$(q^3 - q + 1, q^2, q)$ design with $q = -\mu$.*

Proof. Our remarks above show that μ is a main eigenvalue and that μ is a negative integer. If $H = G - X$ and ν is an eigenvalue of H, we write β_ν for the main angle of H corresponding to ν, and P_ν for the orthogonal projection of $\mathbb{R}^t$ onto the eigenspace of ν. Thus $\beta_\nu = \|P_\nu \mathbf{j}\|/\sqrt{t}$ and $\sum_\nu \beta_\nu^2 = 1$, where the sum is taken over the distinct eigenvalues of H. From Eq.(4) we see that each column of $B^\top$ has precisely μ^2 entries equal to 1, and so $B\mathbf{j} = \mu^2\mathbf{j}$. Since $\mu I = B^\top(\mu I - C)^{-1}B$, we have

$$\mu t = \mu \mathbf{j}^\top \mathbf{j} = \mu^4 \mathbf{j}^\top(\mu I - C)^{-1}\mathbf{j} = \mu^4 \sum_\nu \frac{\mathbf{j}^\top P_\nu \mathbf{j}}{\mu - \nu},$$

whence

$$1 = -\mu^3 \sum_\nu \frac{\beta_\nu^2}{\nu - \mu} \geq -\mu^3 \sum_\nu \frac{\beta_\nu^2}{t - 1 - \mu} = \frac{-\mu^3}{t - 1 - \mu}. \tag{5}$$

The inequality follows. If equality holds in (5) then $t-1$ is the largest eigenvalue of H, while $\beta_\nu = 0$ for all $\nu < t-1$. Hence $C\mathbf{j} = (t-1)\mathbf{j}$ and $H = K_t$. From Eq.(4) we see that $\mathbf{q}_i^\top \mathbf{q}_j = -\mu$ whenever $i \neq j$. Now, there are t neighbourhoods $\Delta_X(i) = \{j \in X : j \sim i\}$ $(i \in V(H))$, each has size μ^2, and any two intersect in $-\mu$ vertices. It now follows from [9, Theorem 1.52] that these neighbourhoods form a symmetric 2-$(-\mu^3 + \mu + 1, \mu^2, -\mu)$ design $\mathcal{D}$. Hence $G = G(\mathcal{D})$, a non-regular graph with spectrum $\mu^{(t)}, -\mu - 1^{(t-1)}, -\mu^3$ (see [12]). Conversely, $G(\mathcal{D})$ satisfies the required conditions. $\square$

As noted in [12], a symmetric 2-$(q^3 - q + 1, q^2, q)$ design exists whenever q is a prime power and $q-1$ is the order of a projective plane (see [5]); moreover there are exactly 78 such designs with $q = 3$ [13]. When $\mu = -2$, the only graph that arises when $t = 7$ is that in Example 2.2 because there is just one symmetric 2-$(7, 4, 2)$ design [5]. In the next section, we give a complete analysis of the case $\mu = -2$.

3 The case $\mu = -2$

Here we assume that the graph G has -2 as an eigenvalue of co-multiplicity t, and that G has an independent star set X for -2 of largest possible size t. By Theorem 2.3, we have $t \geq 7$. If G has components $G_1, \ldots, G_m$ then $X = X_1 \dot\cup \cdots \dot\cup X_m$, where X_i is a star set for μ in G_i $(i = 1, \ldots, m)$. If $G - X_i$ has order t_i then $|X_i| \leq t_i$ by Proposition 2.1. Since $\sum_{i=1}^m |X_i| = t = \sum_{i=1}^m t_i$, we have $|X_i| = t_i$ for each i. Accordingly it suffices to deal with a connected graph G. Since -2 is the least eigenvalue of G, G is either a generalized line graph or an exceptional graph (see [10]). Since -2 is a main eigenvalue, we know that G is not a line graph; in fact, we have:

Lemma 3.1 *G is not a generalized line graph.*

Proof. Suppose by way of contradiction that $G = L(K; a_1, \ldots, a_n)$, where $\sum_{i=1}^n a_i \neq 0$, and that X contains edges from precisely s blossoms in $K(a_1, \ldots, a_n)$. Then X includes at most 2 edges from each of these blossoms, while the remaining edges in X are distributed among $n - s$ vertices of K. Hence $|X| \leq 2s + \frac{1}{2}(n - s)$.

Let m be the number edges in K, so that G has order

$$2t = m + 2\sum_{i=1}^{n} a_i,$$

and by [10, Theorem 2.2.8], -2 has multiplicity

$$t = m - n + \sum_{i=1}^{n} a_i.$$

We deduce that $m = 2n$ and $t = n + \sum_{i=1}^n a_i \geq n + s$. Now we have $n + s \leq |X| \leq 2s + \frac{1}{2}(n - s)$, and so $n \leq s$. Hence $n = s = \sum_{i=1}^n a_i$ and $G = L(K; 1, \ldots, 1)$. Moreover, $G - X = L(K)$. Since the least eigenvalue of $L(K)$ is greater than -2, each component of K is either a tree or an odd unicyclic graph [10, Theorem 2.3.20]. In particular, $m \leq n$, a contradiction. $\square$

It follows that G is an exceptional graph. By [10, Theorem 5.3.1], G has an exceptional star complement H'. By [10, Theorem 2.3.20], H' has order at most 8, and so $t \in \{7, 8\}$. We have seen that when $t = 7$, G is the graph of Example 2.2, and that this graph arises precisely when H is complete.

We now consider the case $t = 8$, where we exploit Eq.(4). If u, v are non-adjacent vertices of H and $w \in V(H) \setminus \{u, v\}$ then $|\Delta_X(u) \cap \Delta_X(w)| = |\Delta_X(v) \cap \Delta_X(w)| = 2$, whence $u \sim w \sim v$. It follows that H can be obtained from K_8 by removing $1, 2, 3$ or 4 independent edges. In particular, each vertex of H has degree 10 or 11 in G, and so X is the unique independent set of size 8 in G.

Let δ be the least degree of a vertex in X, and let u, v be non-adjacent vertices in H. We may take $\Delta_X(u) = \{1, 2, 3, 4\}$ and $\Delta_X(v) = \{5, 6, 7, 8\}$, where vertex 1 has degree δ. To within permutations of 2,3,4 and 5,6,7,8 the following are the possible X-neighbourhoods of the remaining 6 vertices of H:

$\delta = 1$	$\delta = 2$	$\delta = 2$	$\delta = 2$	$\delta = 3$	$\delta = 3$	$\delta = 3$	$\delta = 3$	$\delta = 3$
2356	1258	1258	1268	1256	1256	1256	1256	1257
2378	2356	2356	2356	1278	1278	1278	1278	1358
2457	2378	2378	2378	2357	2357	2357	2357	2356
2468	2457	2457	2458	2368	2368	2458	2368	2378
3458	2468	2468	3457	2458	2458	3456	3456	2458
3467	3467	3458	3468	2467	3456	3478	3478	2467
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)

$\delta = 3$	$\delta = 3$	$\delta = 3$	$\delta = 3$	$\delta = 3$	$\delta = 3$	$\delta = 4$	$\delta = 4$	$\delta = 4$
1268	1268	1258	1267	1267	1258	1256	1256	1256
1358	1367	1356	1356	1378	1356	1278	1357	1357
2356	2356	2357	2357	2357	2456	1357	1458	1467
2378	2378	2456	2456	2456	2478	2468	2367	2358
2458	2458	2478	2478	2478	3458	3456	2468	2468
3457	3457	3458	3458	3458	3467	3478	3478	3478
(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)

Now the permutations (146837)(2)(5), (1)(2)(375)(486), (1653784)(2), (1724368)(5), (1)(253)(467)(8) transform cases (6), (7), (11), (13), (16) to cases (9), (8), (14), (15), (17) respectively. Recall that G is determined by the X-neighbourhoods of vertices in H: the possible graphs are labelled $G_1, \ldots, G_{13}$ in Table 1. They are pairwise non-isomorphic, and most can be distinguished by their degree sequences; where these sequences coincide, it suffices to inspect the intersection numbers $|\Delta_H(j) \cap \Delta_H(k)|$ $(j, k \in X)$ as shown. We summarize our conclusions as follows:

Theorem 3.2 *Let G be a connected graph with -2 as an eigenvalue of co-multiplicity t, and let X be an independent star set for -2 in G. Then $|X| \leq t$, and if $|X| = t$ then either*
(a) *$t = 7$ and $G = G(\mathcal{D})$, where $\mathcal{D}$ is the complement of the Fano plane, or*
(b) *$t = 8$ and G is one of the graphs $G_1, \ldots, G_{13}$ constructed above.*

graph	case(s)	degree sequence	$\lvert\Delta_H(j) \cap \Delta_H(k)\rvert$
G_1	(1)	$11^{(6)}, 10^{(2)}, 5^{(3)}, 4^{(4)}, 1$	
G_2	(2)	$11^{(4)}, 10^{(4)}, 6^{(3)}, 4^{(6)}, 2$	
G_3	(3)	$11^{(6)}, 10^{(2)}, 6, 5^{(2)}, 4^{(2)}, 3^{(2)}, 2$	
G_4	(4)	$11^{(6)}, 10^{(2)}, 5^{(3)}, 4^{(3)}, 3, 2$	
G_5	(5)	$11^{(6)}, 10^{(2)}, 7, 4^{(4)}, 3^{(3)}$	
G_6	(6),(9)	$11^{(4)}, 10^{(4)}, 6, 5, 4^{(3)}, 3^{(3)}$	
G_7	(7),(8)	$11^{(2)}, 10^{(6)}, 5^{(2)}, 4^{(4)}, 3^{(2)}$	$1^{(13)}, 2^{(10)}, 3^{(5)}$
G_8	(10)	$11^{(4)}, 10^{(4)}, 5^{(4)}, 3^{(4)}$	
G_9	(11),(14)	$11^{(2)}, 10^{(6)}, 5^{(2)}, 4^{(4)}, 3^{(2)}$	$0^{(2)}, 1^{(7)}, 2^{(16)}, 3^{(3)}$
G_{10}	(12)	$11^{(6)}, 10^{(2)}, 6, 5, 4^{(3)}, 3^{(3)}$	
G_{11}	(13),(15)	$11^{(2)}, 10^{(6)}, 5^{(2)}, 4^{(4)}, 3^{(2)}$	$0, 1^{(10)}, 2^{(13)}, 3^{(4)}$
G_{12}	(16),(17)	$10^{(8)}, 4^{(8)}$	$1^{(12)}, 2^{(12)}, 3^{(4)}$
G_{13}	(18)	$10^{(8)}, 4^{(8)}$	$0^{(4)}, 2^{(24)}$

Table 1: the graphs from Theorem 3.2(b)

4 Strongly regular graphs

In a 5-cycle, an eigenvalue $\mu = \frac{1}{2}(-1 \pm \sqrt{5})$ has multiplicity 2, while any pair of non-adjacent vertices is a star set for μ. Thus the bound in Proposition 2.1(ii) is sharp for $t = 3$. Here we show how less trivial examples of independent star sets of largest possible size arise from other strongly regular graphs: as we noted in Section 1, it suffices to show that a star set has an independent subset of the appropriate size. Recall that a strongly regular graph G is said to be *primitive* if both G and $\overline{G}$ are connected. We write $m(\mu)$ for the multiplicity of μ in G, and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ for the standard orthonormal basis of $\mathbb{R}^n$. Our starting point is a result from [8]:

Theorem 4.1 [8, Theorem 9.4.1] *Let G be a primitive strongly regular graph of order n with eigenvalues r, μ, λ, where $\lambda < \mu < r$. Let X be an independent set in G. Then*
(i) $|X| \leq m(\lambda)$;
(ii) $|X| \leq n\lambda/(\lambda - r)$;
(iii) *if $|X| = m(\lambda) = n\lambda/(\lambda - r)$ then $G - X$ is strongly regular with eigenvalues $\lambda+\mu, \mu, r+\lambda$ of multiplicities $m(\lambda)-1, m(\mu)-m(\lambda)+1, 1$ respectively.*

We refer to the graphs in part (iii) of this theorem as *coclique-extremal* graphs; examples include the complements of the line graphs $L(K_m)$ ($m \geq 4$). Part (i) says that G has independence number $\alpha(G) \leq t-1$, where t is the co-multiplicity of the (positive) eigenvalue μ. Thus if X is an independent subset of a star set S for μ then

$$|X| \leq \alpha(G_S) \leq \alpha(G) \leq t - 1. \qquad (6)$$

We shall be interested in the case of equality throughout in (6), but first we prove:

Theorem 4.2 *Let G be a primitive strongly regular graph with parameters n, r, e, f and eigenvalues λ, μ, r ($\lambda < \mu < r$). Let X be an independent set of size m in G. Then X is contained in a star set for μ if and only if $m \neq 1 + r(-\lambda - 1)/f$.*

Proof. If G has adjacency matrix A then the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$ has matrix

$$P = \frac{1}{(\mu - r)(\mu - \lambda)}(A - rI)(A - \lambda I).$$

Thus the principal submatrix of $(\mu - r)(\mu - \lambda)P$ determined by X is the matrix $M = f(J-I)+rI+r\lambda I$. Now X is contained in a star set for μ if and only if the vectors $P\mathbf{e}_i$ ($i \in X$) are linearly independent [11, Proposition 5.1]. Since P is symmetric, the columns $P\mathbf{e}_i$ ($i \in X$) are linearly independent if and only if M is invertible [11, Lemma 5.1.5]. The eigenvalues of M are $f(m-1) + r(1+\lambda)$ and $r(1+\lambda) - f$ (a negative eigenvalue of multiplicity $m - 1$). Therefore X is contained in a star set for μ if and only if $m \neq 1 + r(-\lambda - 1)/f$. $\square$

Invoking Theorem 4.2 with $m + 1$ and $m - 1$ in place of m, we deduce:

Corollary 4.3 *Let G be a primitive strongly regular graph with parameters n, r, e, f and eigenvalues λ, μ, r ($\lambda < \mu < r$). Let X be an independent set of size m in G. If X is not contained in a star set for μ, then $m = 1 + r(-\lambda - 1)/f$, X is a maximal independent set and every proper subset of X is contained in a star set for μ.*

For the next result we invoke Theorem 4.2 in the case that m takes the largest possible value.

Corollary 4.4 *Let G be a primitive strongly regular graph with a positive eigenvalue μ of co-multiplicity t, so that $\alpha(G) \leq t - 1$. Suppose that G contains an independent set X of size $t - 1$. Then X is contained in a star set for μ if and only if G is not coclique-extremal.*

Proof. Again suppose that G has parameters n, r, e, f. We have $\mu \neq r$ for otherwise $|X| = n - 2$ and G is a 4-cycle. By Theorem 4.2, X is not contained in a star set for μ if and only if

$$m(\lambda) = 1 + \frac{r}{f}(-1 - \lambda), \tag{7}$$

where λ is the negative eigenvalue of G. By Theorem 4.1, X is coclique-extremal if and only if

$$m(\lambda) = \frac{n\lambda}{\lambda - r}. \tag{8}$$

Now in any primitive strongly regular graph, we have [11, Theorem 3.6,4]:

$$f = r + \lambda\mu, \quad n = \frac{(r - \mu)(r - \lambda)}{r + \lambda\mu}.$$

It follows that

$$1 + \frac{r}{f}(-1 - \lambda) = \frac{\lambda(\mu - r)}{r + \lambda\mu} = \frac{n\lambda}{\lambda - r}.$$

Hence conditions (7) and (8) are equivalent, and the result follows. □

Examples 4.5 (i) In the Petersen graph $G = \overline{L(K_5)}$, a largest independent set X has size 4, and for any such X we have $G - X = 3K_2$. Thus X is a star set for the eigenvalue -2. By Corollary 4.4, it is not contained in a star set for the eigenvalue 1 because G is coclique-extremal.

(ii) Let Sch_{10} denote the unique strongly regular graph with parameters $27, 10, 1, 5$ and spectrum $-5^{(6)}, 1^{(20)}, 10$ [9, p.22]: in the literature, both Sch_{10} and its complement are referred to as the Schläfli graph. We write McL_{112} for the McLaughlin graph, the unique strongly regular graph with parameters $275, 112, 30, 56$ and spectrum $-28^{(22)}, 2^{(252)}, 112$ [16]. Both Sch_{10} and McL_{112} are extremal strongly regular graphs but they are not coclique-extremal. As noted in [17], Sch_{10} has an independent set X_1 of size $6 = m(-5)$, and McL_{112} has an independent set X_2 of size $22 = m(-28)$. By Corollary 4.4, X_1 lies in a star set for 1, and X_2 lies in a star set for 2. We deduce that the bound of Proposition 2.1(ii) is sharp for $t = 7$ and $t = 23$. □

5 A connection with codes

Here we confine our investigations to star complements of a specific type. We have seen that if X is an independent star set for the non-zero eigenvalue μ of G, and if $G - X \cong K_t$, then t is a sharp upper bound for $|X|$. As we point out below, whenever G has a star set S for μ such that $G - S \cong K_t$ $(t > 1)$, μ is necessarily a main eigenvalue. However, for a non-main eigenvalue μ, the case $G - S \cong sK_t$ $(s > 1, t > 1)$ turns out to be of interest in relation to codes. (In this situation, μ has co-multiplicity st.)

We first assume that μ is a non-zero eigenvalue of G, and that a star complement H for μ has the form $G - S = H_1 \dot\cup \cdots \dot\cup H_s$, where each H_i is complete of order t $(s \geq 1, t > 1)$. Thus $\mu \neq t-1$ or -1. For distinct vertices $u, v \in X$, we denote the characteristic vectors of $\Delta_{H_i}(u), \Delta_{H_i}(v)$ by $\mathbf{u}_i, \mathbf{v}_i$ respectively, and we write $u_i = \mathbf{j}^\top \mathbf{u}_i$ $(= \mathbf{u_i}^\top \mathbf{u}_i)$, $v_i = \mathbf{j}^\top \mathbf{v}_i$ $(= \mathbf{v_i}^\top \mathbf{v}_i)$ $(i = 1, \ldots, s)$.

We use the notation of Theorem 1.1. The matrix $(\mu I - C)^{-1}$ is block diagonal, with each of the s diagonal blocks equal to

$$\frac{1}{\mu+1} I - \frac{1}{(\mu+1)(t-\mu-1)} J,$$

where I, J are the identity and all-one matrices of size $t \times t$. From Eq. (2) we have

$$\frac{1}{\mu+1} \sum_{i=1}^{s} u_i - \frac{1}{(\mu+1)(t-\mu-1)} \sum_{i=1}^{s} u_i^2 = \mu \tag{9}$$

(with a similar relation for the v_i) and

$$\frac{1}{\mu+1} \sum_{i=1}^{s} \mathbf{u}_i^\top \mathbf{v}_i - \frac{1}{(\mu+1)(t-\mu-1)} \sum_{i=1}^{s} u_i v_i = \begin{cases} -1 \text{ if } u \sim v \\ \ \ 0 \text{ if } u \not\sim v. \end{cases} \tag{10}$$

Lemma 5.1 *If μ is a non-main eigenvalue of G then*

$$\sum_{i=1}^{s} u_i = t - \mu - 1, \qquad \sum_{i=1}^{s} u_i^2 = (t-\mu-1)(t-(\mu+1)^2). \tag{11}$$

Proof. From Eq. (3) we have

$$\frac{1}{\mu+1} \sum_{i=1}^{s} u_i - \frac{1}{(\mu+1)(t-\mu-1)} \sum_{i=1}^{s} u_i t = -1,$$

whence $\sum_{i=1}^{s} u_i = t - \mu - 1$. Substituting for $\sum_{i=1}^{s} u_i$ in Eq.(9), we obtain the second assertion in (11). □

Henceforth we assume μ is non-main. If $s = 1$ then $u_1 \neq 0$ (since $\mu \neq 0$); in this case, Eq.(11) yields $t - (\mu+1) = u_1 = t - (\mu+1)^2$, whence $\mu = -1$, a contradiction. Hence $s > 1$.

It follows from Eq.(11) that μ is an integer and $t \geq \mu^2 + 2\mu + 2$. The connection with codes arises when G is connected and $t = \mu^2 + 2\mu + 2$: this is the case we address here.

From Eq.(11) we have $\sum_{i=1}^{s}(u_i^2 - u_i) = 0$, and so each u_i is 0 or 1; moreover, exactly $t - \mu - 1$ of the u_i are equal to 1. The same is true of the v_i, and so $\sum_{i=1}^{s} u_i v_i \leq t - \mu - 1$. By Eq. (10), $t - \mu - 1$ divides $\sum_{i=1}^{s} u_i v_i$, and so $\sum_{i=1}^{s} u_i v_i$ is either $t - \mu - 1$ or 0. In the latter case, $\sum_{i=1}^{s} \mathbf{u}_i^\top \mathbf{v}_i = 0$ and so $u \not\sim v$; moreover, u_i or v_i is zero for each i. Since G is connected, it follows that $s = t - \mu - 1 = \mu^2 + \mu + 1$, and $\sum_{i=1}^{s} u_i v_i = t - \mu - 1$ for all $u, v \in X$.

We now label the vertices of each K_t by $0, 1, 2, \ldots, t-1$, so that each neighbourhood $\Delta_H(u)$ $(u \in S)$ can be specified by a t-ary codeword $\mathbf{c}_u$ of length s. In this situation we say that S is *represented* by the code $\{\mathbf{c}_u : u \in S\}$. The (Hamming) distance between codewords $\mathbf{c}_u, \mathbf{c}_v$ is denoted by $h(\mathbf{c}_u, \mathbf{c}_v)$.

Lemma 5.2 *For distinct vertices $u, v \in S$, we have*

$$h(\mathbf{c}_u, \mathbf{c}_v) = \begin{cases} \mu^2 + \mu & \text{if } u \not\sim v \\ (\mu+1)^2 & \text{if } u \sim v. \end{cases} \tag{12}$$

Moreover, $\mu < 0$ when S is not an independent set.

Proof. From Eq. (10) we have

$$\sum_{i=1}^{s} \mathbf{u}_i^\top \mathbf{v}_i = \begin{cases} 1 & \text{if } u \not\sim v \\ -\mu & \text{if } u \sim v. \end{cases} \tag{13}$$

Thus $\mu < 0$ when S contains adjacent vertices, while Eq. (7) follows from the observation that $h(\mathbf{c}_u, \mathbf{c}_v) = s - |\Delta_H(u) \cap \Delta_H(v)| = \mu^2 + \mu + 1 - \sum_{i=1}^{s} \mathbf{u}_i^\top \mathbf{v}_i$. □

An $(n, M, d)_q$ code is a q-ary code of length n, cardinality M and minimum distance at least d. As usual we write $A_q(n, d)$ for the maximum possible number of codewords in an $(n, M, d)_q$ code. It follows from Lemma 5.2 that if $|S| = k$ then G can be constructed from H by adding k vertices represented by a $(\mu^2 + \mu + 1, k, \mu^2 + \mu)_t$ code or an appropriate $(\mu^2 + \mu + 1, k, (\mu+1)^2)_t$ code. Moreover, existence of an independent star set X of size k is equivalent to the existence of a $(\mu^2 + \mu + 1, k, \mu^2 + \mu)_t$ code. Thus we have the following:

Theorem 5.3 *Let G be a connected graph with an independent star set X for the non-zero non-main eigenvalue μ. If $G - X \cong sK_t$ $(s > 1, t > 1)$ then μ is an integer and $t \geq \mu^2 + 2\mu + 2$; moreover, if $t = \mu^2 + 2\mu + 2$ then $s = \mu^2 + \mu + 1$ and $|X| \leq A_t(s, s-1)$.*

As observed in [6], a good upper bound for $A_t(s, s-1)$ can be found from the following result:

Theorem 5.4 [7, Theorem 3] *If there exists an $(n, M, d)_q$ code then*

$$M(M-1)d \leq 2n \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} M_i M_j,$$

where $M_i = \lfloor (M+i)/q \rfloor$.

We conclude with examples which illustrate the extremal case $t = \mu^2 + 2\mu + 2$ of Theorem 5.3. As usual, we write H for $G - X$.

Examples 5.5 (i) When $\mu = -2$ we have $s = 3$, $t = 2$ and $A_2(3, 2) = 4$. The Petersen graph can be constructed from the star complement $3K_2$ by adding 4 (independent) vertices represented by the code $\{000, 011, 101, 110\}$.

(ii) When $\mu = 1$, we have $s = 3$, $t = 5$, $H = 3K_5$ and $A_5(3, 2) = 5$. When G is maximal, the possible codes are (without loss of generality) $\{000, 011, 101, 110\}$ and $\{000, 011, 022, 033, 044\}$. These determine graphs of order 19 and 20 with an independent star set for 1 of size 4 and 5 respectively.

(iii) When $\mu = -3$, we have $s = 7$, $t = 5$, $H = 7K_5$ and $A_5(7, 6) = 15$ (see [6]). Indeed $A_5(7, 6) \leq 15$ by Theorem 5.4, while $A_5(7, 6) \geq 15$ because a $(7, 15, 6)_5$ code can be constructed from a resolution of a 2-$(15, 3, 1)$ design, that is, a Kirkman triple system on 15 points; in fact, every $(7, 15, 6)_5$ code arises in this way [18], and there are exactly seven essentially different resolutions of a 2-$(15, 3, 1)$ design [15, Table 6.15]. The first such design in [14, Table 17.2] gives the following code, which determines a graph of order 50 with an independent star set for -3 of size 15:

0421010	0040441	0102234	1032112	1143023
1414201	2111342	2204413	2323131	3241104
3312420	3430333	4003300	4220222	4334044

(iv) When $\mu = -4$, we have $H = 13K_{10}$ and then $|X| \leq A_{10}(13, 12) \leq 40$ by Theorems 5.3 and 5.4. $\Box$

Acknowledgement The author is indebted to P. R. J. Östergård for references [15] and [18].

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