ON GRAPHS WITH AN EIGENVALUE OF MAXIMAL MULTIPLICITY


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**Abstract**

Let $G$ be a graph of order $n$ with an eigenvalue $\mu \neq -1, 0$ of multiplicity $k < n-2$. It is known that $k \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$, equivalently $k \leq \frac{1}{2}t(t-1)$, where $t = n - k > 2$. The only known examples with $k = \frac{1}{2}t(t-1)$ are $3K_2$ (with $n = 6$, $\mu = 1$, $k = 3$) and the maximal exceptional graph $G_{36}$ (with $n = 36$, $\mu = -2$, $k = 28$). We show that no other example can be constructed from a strongly regular graph in the same way as $G_{36}$ is constructed from the line graph $L(K_9)$.



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# 1 Introduction

Let $G$ be a graph of order $n$ with an eigenvalue $\mu \neq -1, 0$ of multiplicity $k < n-2$. It was shown in [1] that $k \leq n + \frac{1}{2} - \sqrt{2n + \frac{1}{4}}$, equivalently $k \leq \frac{1}{2}t(t-1)$, where $t = n-k > 2$. The only known examples with $k = \frac{1}{2}t(t-1) > 1$ are $3K_2$, with spectrum $-1^{(3)}, 1^{(3)}$, and the unique maximal exceptional graph of order 36, with spectrum $21, 5^{(7)}, -2^{(28)}$. The latter graph is described in [3, Chapter 6] and [4, Example 5.2.6(a)]; it is denoted here by $G_{36}$. After a decade, it remains a problem to determine all the graphs with $k = \frac{1}{2}t(t-1)$. The restricted question, of similar standing, is whether further examples can be constructed from a strongly regular graph in the same way that $G_{36}$ is constructed from the line graph $L(K_9)$. Here we answer this question in the negative.

To describe the construction we recall some notation and terminology from [4]. For a subset $X$ of the vertex set $V(G)$, we write $\overline{X}$ for $V(G) \setminus X$, $G - X$ for the subgraph of $G$ induced by $\overline{X}$, and $G_X$ for the graph obtained from $G$ by switching with respect to $X$. We say that $X$ is a *star set* for $\mu$ if $|X| = k$ and $\mu$ is not an eigenvalue of $G - X$. Our main result is the following.

**Theorem 1.1.** *Let $G$ be a graph of order $\frac{1}{2}t(t+1)$ $(t > 2)$ with an eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity $\frac{1}{2}t(t-1)$. Suppose that $G$ has a star set $X$ for $\mu$ such that (i) $X \dot\cup \overline{X}$ is an equitable partition of $G$, (ii) $G_X$ is a strongly regular graph. Then $t = 8$, $\mu = -2$ and $G = G_{36}$.*

Note that, in the situation of Theorem 1.1, $X \dot\cup \overline{X}$ is also an equitable partition of $G_X$. To construct $G_{36}$, we take $G_X = L(K_9)$ and choose $X$ so that $X$ induces $L(K_8)$ and $\overline{X}$ induces $K_8$.

# 2 Prerequisites

If $X$ is a star set for $\mu$ in $G$, then $G - X$ is said to be a *star complement* for $\mu$ in $G$. Star sets and star complements exist for any eigenvalue of any graph, and their basic properties are described in [4, Chapter 5]. In particular, we shall require the following result.

**Theorem 1.1** [4, Theorem 5.1.7] *Let $X$ be a set of $k$ vertices in the graph $G$ and suppose that $G$ has adjacency matrix $A = \begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$, where $A_X$ is the adjacency matrix of the subgraph induced by $X$. Then $X$ is a star set for $\mu$ in $G$ if and only if $\mu$ is not an eigenvalue of $C$ and*

$$\mu I - A_X = B^T(\mu I - C)^{-1}B. \tag{1}$$

Suppose that $X$ is a star set, and let $H = G - X$. In Theorem 1.1, $k$ is the multiplicity of $\mu$, and $C$ is the adjacency matrix of $H$. Also, the columns of $B$ are the characteristic vectors of the $H$-neighbourhoods

$$\Delta_H(u) = \{v \in V(H) : u \sim v\} \ (u \in X),$$

where we write '$u \sim v$' to mean that vertices $u, v$ are adjacent in $G$. Equation (1) shows that any graph is determined by an eigenvalue $\mu$, a star complement $H = G - X$ and the $H$-neighbourhoods of vertices in $X$. When $G - X$ is complete, we obtain the following by equating diagonal entries in Equation (1).

**Lemma 2.2.** *Suppose that $X$ is a star set for $\mu$ in the graph $G$. Let $H = G - X$, $u \in X$. If $H = K_t$ $(t > 2)$ and $|\Delta_H(u)| = a$ then*

$$a^2 - (t - \mu - 1)a + \mu(\mu + 1)(t - \mu - 1) = 0.$$

In the general case, we let $|V(H)| = t > 2$ and define a bilinear form on $\mathbb{R}^t$ by

$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle \ = \mathbf{x}^\top(\mu I - C)^{-1}\mathbf{y} \ \ (\mathbf{x}, \mathbf{y} \in \mathbb{R}^t).$$

We let $V(G) = \{1, 2, \ldots, n\}$ and write $S = (B|C - \mu I)$, with columns $\mathbf{s}_u$ $(u = 1, \ldots, n)$. Let $\mathcal{Q}_t$ denote the space of homogeneous quadratic functions on $\mathbb{R}^t$. We define $F_1, \ldots, F_n \in \mathcal{Q}_t$ by

$$F_u(\mathbf{x}) = \langle\langle \mathbf{s}_u, \mathbf{x} \rangle\rangle^2 \qquad (\mathbf{x} \in \mathbb{R}^t).$$

**Lemma 2.3.** [1, Lemma 2.2] *If $t > 2$ and $\mu \neq -1$ or $0$, the functions $F_1, \ldots, F_n$ are linearly independent.*

Since $\dim \mathcal{Q}_t = \binom{t}{2} + t$, we deduce that $n \leq \binom{t}{2} + t$, equivalently $k \leq \binom{t}{2}$. The following result enables us to dispose of the regular graphs for which this bound is attained.

**Theorem 2.4.** [1, Theorem 3.1] *Let $G$ be an $r$-regular graph $G$ of order $n$ with $\mu$ as an eigenvalue of multiplicity $k$. If $\mu \notin \{-1, 0, r\}$ and $t = n - k > 2$ then $k \leq \binom{t}{2} - 1$.*

**Corollary 2.5.** *If $G$ is a regular graph of order $\frac{1}{2}t(t + 1)$ $(t > 2)$ with an eigenvalue $\mu \notin \{-1, 0\}$ of multiplicity $\frac{1}{2}t(t - 1)$ then $t = 3$, $\mu = 1$ and $G = 3K_2$.*

**Proof.** If $G$ is $r$-regular then $\mu = r$ by Theorem 2.4, and so $G$ has $\frac{1}{2}t(t-1)$ components, each with $\mu$ as a simple eigenvalue (cf. [4, Corollary 1.3.8]). It follows that $t \geq \frac{1}{2}t(t - 1)$, and hence that $t = 3$, $G = 3K_2$, $\mu = 1$. □

Next, using Equation (1), we see that

$$\mu I - A = S^\top(\mu I - C)^{-1}S,$$

and so, for all vertices $u, v$ of $G$,

$$\langle\!\langle \mathbf{s}_u, \mathbf{s}_v \rangle\!\rangle = \begin{cases} \mu & \text{if } u = v \\ -1 & \text{if } u \sim v \\ 0 & \text{otherwise} \end{cases}. \tag{2}$$

It follows that if $\mu \notin \{-1, 0\}$ then the $H$-neighbourhoods $\Delta_H(u)$ $(u \in X)$ are distinct and non-empty. When $k = \binom{t}{2}$, our objective will be to show that, under suitable conditions, the $H$-neighbourhoods form a tight 4-design, that is, a design which satisfies the following conditions with $s = 2$.

**Theorem 2.6.** [2, Theorem 1.52] *Let $\mathcal{B}$ be a collection of $a$-subsets of the $t$-set $V$, where $2s \le a \le t - s$. Then any two of the following conditions imply the third.*

*(a) $(V, \mathcal{B})$ is a $2s$-design;*

*(b) there are precisely $s$ values for the numbers $|B \cap B'|$, where $B, B'$ are distinct sets in $\mathcal{B}$;*

*(c) $|\mathcal{B}| = \binom{t}{s}$.*

Finally we can exploit the fact that tight 4-designs are extremely rare:

**Theorem 2.7.** [2, Theorem 1.54] *Let $\mathcal{D}$ be a tight $4$-$(t, a, l)$ design with $4 \le a < t - 2$. Then either $\mathcal{D}$ or its complement $\overline{\mathcal{D}}$ is the unique $4$-$(23, 7, 1)$ design.*

# 3 Proof of the main result

We retain the notation of the previous sections. Additionally we suppose that $k = \frac{1}{2}t(t-1)$ $(t > 2)$, and that the star set $X$ for $\mu \neq -1, 0$ is such that (i) $X \,\dot\cup\, \overline{X}$ is an equitable partition of $G$, (ii) $G_X$ is strongly regular with parameters $(n, r, e, f)$ $(0 < r < n-1)$. We show first that $t \neq 3$ by inspecting the strongly regular graphs of order 6. If $G_X = 2K_3$ or $\overline{2K_3}$ then there is no suitable bipartition $X \,\dot\cup\, \overline{X}$. If $G_X = 3K_2$ then $G - X = 3K_1$ and $G = C_6$, while if $G = \overline{3K_2}$ then $G - X = K_3$ and $G = \overline{C_6}$. In both cases $G$ has no eigenvalue of multiplicity 3. Hence $t > 3$ and $k > \frac{1}{2}n$. It follows that $\mu$ is an integer, for otherwise $\mu$ has an algebraic conjugate which is a second eigenvalue of multiplicity $k$.

The partition $X \,\dot\cup\, \overline{X}$ determines divisors of $G$ and $G_X$, and we denote the corresponding divisor matrices by

$$D = \begin{pmatrix} p & a \\ b & q \end{pmatrix}, \quad D^* = \begin{pmatrix} p & t-a \\ k-b & q \end{pmatrix},$$

respectively. Note that $|\Delta_H(u)| = a$ for all $u \in X$, and that $1 < a < t-1$. In what follows, we write $\mathbf{j}$ for an all-1 vector (with length determined by context), and $A^*$ for the adjacency matrix of $G_X$. Additionally, $\mathcal{E}$ denotes an eigenspace of $G$ and $\mathcal{E}^*$ denotes an eigenspace of $G_X$.

**Lemma 3.1.** *There exist integers $\lambda, \rho$ such that $G_X$ has spectrum $r, \lambda^{(t)}, \mu^{(k-1)}$ and $G$ has spectrum $\rho, \lambda^{(t-1)}, \mu^{(k)}$.*

**Proof.** Let $\mathcal{V}$ be the subspace of $\mathbb{R}^n$ spanned by the characteristic vectors of $X$ and $\overline{X}$, and let $\mathcal{W} = \mathcal{V}^\perp$. Note that for any eigenvalue $\nu$, $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathcal{W} \cap \mathcal{E}(\nu)$ if and only if $\begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix} \in \mathcal{W} \cap \mathcal{E}^*(\nu)$. The graph $G$ has linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{V}$ with corresponding eigenvalues those of $D$, while $G_X$ has linearly independent eigenvectors $\mathbf{x}_1^*, \mathbf{x}_2^* \in \mathcal{V}$ with corresponding eigenvalues those of $D^*$. Moreover, if $\rho$ is the largest eigenvalue of $G$ then we may take $A\mathbf{x}_1 = \rho\mathbf{x}_1$, $A^*\mathbf{x}_1^* = r\mathbf{x}_1^*, \mathbf{x}_1^* = \mathbf{j}$ (cf. [4, Theorem 3.9.9]). Since $\mathcal{E}(\mu)$ and $\mathcal{W}$ are subspaces of $\mathbf{x}_1^\perp$, we have $\dim(\mathcal{E}(\mu) \cap \mathcal{W}) \geq k-1$. On the other hand, $\dim \mathcal{E}^*(\mu) \leq k-1$ by Lemma 2.3, and so we deduce that $\mathcal{E}^*(\mu) \subseteq \mathcal{W}$, $\dim(\mathcal{E}(\mu) \cap \mathcal{W}) = k-1$, and $A\mathbf{x}_2 = \mu\mathbf{x}_2$. Let $A^*\mathbf{x}_2^* = \lambda\mathbf{x}_2^*$. Then $\mu \neq \lambda = p+q-r \in \mathbb{Z}$ and $G_X$ has spectrum $r, \lambda^{(t)}, \mu^{(k-1)}$ (cf. [4, Section 3.6]). Note that $\lambda \neq r$ for otherwise $G_X = (t+1)K_{r+1}$ and $\mu = -1$, contrary to assumption. We deduce that $G$ has spectrum $\rho, \lambda^{(t-1)}, \mu^{(k)}$. Finally, $\rho \in \mathbb{Z}$ because $\rho = p+q-\mu$. □

When $G = G_{36}$ and $G_X = L(K_9)$, we have $\mu = -2,\ t = 8,\ k = 28,\ r = 14,\ \rho = 21,\ \lambda = 5,\ p = 12,\ q = 7,\ a = 6,\ b = 21$.

**Lemma 3.2.** *The matrix $\mu^2 I + A$ is invertible.*

**Proof.** Since $\mu^2 \notin \{-\rho, -\mu\}$, it suffices to show that $\mu^2 \neq -\lambda$. Now the multiplicities of $\lambda$ and $\mu$ in the strongly regular graph $G_X$ are given by

$$m(\lambda) = \frac{r(r-\mu)(\mu+1)}{(r+\lambda\mu)(\mu-\lambda)}, \quad m(\mu) = \frac{r(r-\lambda)(\lambda+1)}{(r+\lambda\mu)(\lambda-\mu)},$$

formulae which follow from [4, Theorems 3.6.4 and 3.6.5]. Suppose by way of contradiction that $\lambda = -\mu^2$. Then $\mu > 0$. Since $m(\lambda) = t$ and $m(\mu) = k-1 = \frac{1}{2}(t+1)(t-2)$, we have:

$$\frac{(t+1)(t-2)}{2t} = \frac{m(\mu)}{m(-\mu^2)} = \frac{(r+\mu^2)(\mu-1)}{r-\mu}. \tag{3}$$

Let $\theta = (r-\mu)/t$. Then

$$0 = \mathrm{tr}(A^*) = \mu + \theta t + t(-\mu^2) + \frac{1}{2}(t+1)(t-2)\mu,$$

whence $\theta = \mu^2 - \frac{1}{2}(t-1)\mu$. Substituting $\mu + \mu^2 t - \frac{1}{2}\mu t(t-1)$ for $r$ in Equation (3), and dividing by $\mu t(t+1)$, we obtain:

$$(t-2\mu)(t-2\mu-1) = 2(1-\mu). \tag{4}$$

Since $\mu \in I\!N$, the left hand side of (4) is non-negative, while the right hand side is non-positive. We conclude that $\mu = 1$ and $t \in \{2, 3\}$, a contradiction. $\square$

We are now in a position to prove the following.

**Lemma 3.3.** *If $4 \le a \le t-2$, the $H$-neighbourhoods $\Delta_H(u)$ $(u \in X)$ form a tight 4-design.*

**Proof.** By Lemma 2.3, the functions $\langle\langle \mathbf{s}_u, \mathbf{x} \rangle\rangle^2$ $(u \in X)$ form a basis for $\mathcal{Q}_t$. Let

$$\langle\langle \mathbf{x}, \mathbf{x} \rangle\rangle = \sum_{u=1}^{n} \gamma_u \langle\langle \mathbf{s}_u, \mathbf{x} \rangle\rangle^2, \tag{5}$$

and write $\mathbf{c} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^\top$. From Equation (2) we have

$$\mu = \langle\langle \mathbf{s}_i, \mathbf{s}_i \rangle\rangle = \mu^2 \gamma_i + \sum_{u \sim i} \gamma_u \qquad (i = 1, 2, \ldots, n),$$

whence $(\mu^2 I + A)\mathbf{c} = \mu \mathbf{j}$. In view of Lemma 3.2, we have $\mathbf{c} = (\mu^2 I + A)^{-1} \mu \mathbf{j}$. In the notation of Lemma 3.1, we have $\mathbf{j} \in \mathcal{V}$, while $\mathcal{V}$ is $A$-invariant since the eigenvectors $\mathbf{x}_1^*, \mathbf{x}_2^*$ form a basis for $\mathcal{V}$. It follows that there exist $\xi, \eta \in I\!R$ such that $\mathbf{c} = \begin{pmatrix} \xi \mathbf{j} \\ \eta \mathbf{j} \end{pmatrix}$.

We extend notation in a natural way, so that for example $\Delta_H^*(u)$ denotes the set of vertices in $\overline{X}$ adjacent to $u$ in $G_X$. For $i, j \in X$, let $r_{ij} = |\Delta_X(i) \cap \Delta_X(j)|$, $s_{ij} = |\Delta_H(i) \cap \Delta_H(j)|$ and $t_{ij} = |\Delta_H^*(i) \cap \Delta_H^*(j)|$. Note that $r_{ij} = |\Delta_X^*(i) \cap \Delta_X^*(j)|$ and $t_{ij} = t - 2a + s_{ij}$. Since $r_{ij} + t_{ij}$ is the number of common neighbours of $i$ and $j$ in $G_X$, we have

$$r_{ij} + s_{ij} = \begin{cases} 2a - t + e & \text{if } i \sim j \\ 2a - t + f & \text{if } i \not\sim j \end{cases}. \tag{6}$$

Since $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \frac{1}{4}(\langle\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y} \rangle\rangle - \langle\langle \mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{y} \rangle\rangle)$, Equation (5) yields

$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \sum_{u=1}^{n} \gamma_u \langle\langle \mathbf{s}_u, \mathbf{x} \rangle\rangle \langle\langle \mathbf{s}_u, \mathbf{y} \rangle\rangle.$$

Setting $\mathbf{x} = \mathbf{s}_i$, $\mathbf{y} = \mathbf{s}_j$ we obtain:

$$-1 = \xi(r_{ij} - 2\mu) + \eta s_{ij} \ \text{ if } i \sim j, \qquad 0 = \xi r_{ij} + \eta s_{ij} \ \text{ if } i \not\sim j. \tag{7}$$

Thus from (6) and (7) we obtain two sets of simultaneous equations in $r_{ij}$ and $s_{ij}$:

$$\left.\begin{array}{l} r_{ij} + s_{ij} = 2a - t + e \\ \xi r_{ij} + \eta s_{ij} = 2\mu\xi - 1 \end{array}\right\} \text{ if } i \sim j, \qquad \left.\begin{array}{l} r_{ij} + s_{ij} = 2a - t + f \\ \xi r_{ij} + \eta s_{ij} = 0 \end{array}\right\} \text{ if } i \not\sim j.$$

Now $\xi \neq \eta$ for otherwise $\mathbf{j}$ is an eigenvector of $A$, and $G$ is regular, contradicting Corollary 2.5. Therefore each set of simultaneous equations has a unique solution; in particular, there exist integers $e', f'$ such that

$$|\Delta_H(i) \cap \Delta_H(j)| = \begin{cases} e' & \text{if } i \sim j, \\ f' & \text{if } i \not\sim j. \end{cases}$$

We have $e' \neq f'$, for otherwise $|X| \leq t$ [2, Theorem 1.51]. Thus if $4 \leq a \leq t-2$ then by Theorem 2.7 the $H$-neigbourhoods $\Delta_H(u)$ $(u \in X)$ form a tight 4-design. □

In view of Theorem 2.7, it remains to consider four cases: (a) $t = 23$ and $a \in \{7, 16\}$, (b) $a = 3$, (c) $a = 2$, (d) $a = t - 2$.

*Case (a).* In this case we have $n = 276$, $|X| = 253$, $|\overline{X}| = 23$ and either $a = 7$ or $a = 16$. If $a = 7$ then $D^* = \begin{pmatrix} r-16 & 16 \\ 176 & r-176 \end{pmatrix}$, whence $176 \leq r \leq 198$. If $a = 16$ then $D^* = \begin{pmatrix} r-7 & 7 \\ 77 & r-77 \end{pmatrix}$, whence $77 \leq r \leq 99$. For these values of $r$, there is no strongly regular graph of order 276 and degree $r$; see for example Brouwer's list of feasible parameters at `http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html`.

*Case (b): $a = 3$.* Let $\mathcal{D} = \{\Delta_H(u) : u \in X\}$. If $t \geq 7$ then $\overline{\mathcal{D}}$ is a tight 4-design and Theorem 2.7 is contradicted. If $t < 7$ then $t \in \{5, 6\}$ and the multiplicity of $\mu$ in $G_X$ is $9, 14$ respectively. If $t = 5$ then either $G_X = L(K_6)$ with $\mu = -2$ or $G_X = \overline{L(K_6)}$ with $\mu = 1$. In the former case, $D^* = \begin{pmatrix} 6 & 2 \\ 4 & 4 \end{pmatrix}$, and so $H = K_5$; but the graph obtained from $K_5$ by adding a vertex of degree 3 does not have $-2$ as an eigenvalue (see Lemma 2.2). In the latter case, $D^* = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}$, and so $H$ is a 5-cycle; but then not all 3-subsets of $\overline{X}$ can be $H$-neighbourhoods in $G$ [4, Example 5.2.3]. Now suppose that $t = 6$. Then $k = 15$ and we have $b = ka/t = 45/6$, a contradiction.

*Case (c): $a = 2$.* Here the $H$-neighbourhoods in $G$ are all the 2-subsets of $\overline{X}$, and their intersection numbers are necessarily 0 and 1. We have

$$D^* = \begin{pmatrix} r - t + 2 & t - 2 \\ \frac{1}{2}(t-1)(t-2) & r - \frac{1}{2}(t-1)(t-2) \end{pmatrix},$$

whence $\lambda = r - \frac{1}{2}(t-2)(t+1)$. Now

$$0 = \mathrm{tr}(A^*) = r + t(r - \tfrac{1}{2}(t-2)(t+1)) + \tfrac{1}{2}(t+1)(t-2)\mu,$$

whence

$$\mu = t - \frac{2r}{t-2}. \tag{8}$$

A 2-subset of $\overline{X}$ intersects precisely $2t-4$ other 2-subsets of $\overline{X}$. Hence if $(e', f') = (1, 0)$ then each vertex in $X$ has $2t-4$ neighbours in $X$, and so $2t-4 = r-t+2$. Thus $r = 3(t-2)$, and we see from Equation (8) that $\mu = t-6$. Now $r \geq \frac{1}{2}(t-1)(t-2)$ and so $t \leq 7$. Since $\mu \neq -1$ or 0, we have $t = 4$ or $t = 7$. If $t = 4$ then

$$D^* = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} = D,$$

whence $G$ is 6-regular, a contradiction. If $t = 7$ then $r = 15$, $\mu = 1$ and $\overline{X}$ is an independent set. Equating diagonal entries in Equation (1), we find that $a = 1$, a contradiction.

If $(e', f') = (0, 1)$ then each vertex in $X$ has $2t-4$ non-neighbours in $X$, and so $2t-4 = \frac{1}{2}t(t-1) - 1 - (r-t+2)$. We find that $r = \frac{1}{2}(t-1)(t-2)$, $\mu = 1$ and $\overline{X}$ is independent, leading to the same contradiction as above.

*Case (d): $a = t-2$.* In this case we have $D^* = \begin{pmatrix} r-2 & 2 \\ t-1 & r-t+1 \end{pmatrix}$, whence $\lambda = r-t-1$. Now

$$0 = \mathrm{tr}(A^*) = r + t(r-t-1) + \frac{1}{2}(t+1)(t-2)\mu,$$

and so

$$\mu = \frac{2(t-r)}{t-2}.$$

For distinct $u, v \in X$, let

$$|\Delta_H^*(u) \cap \Delta_H^*(v)| = \begin{cases} e^* \text{ if } u \sim v, \\ f^* \text{ if } u \not\sim v \end{cases},$$

so that $\{e^*, f^*\} = \{0, 1\}$.

If $(e^*, f^*) = (0, 1)$ then each vertex of $X$ has $2t-4$ non-neighbours in $X$, and so $r-2 = \binom{t}{2} - (2t-4) - 1$, whence $r = \frac{1}{2}(t^2 - 5t + 10)$ and $\mu = 5-t$. Since $r-t+1 \leq t-1$, we have $(t-2)(t-7) \leq 0$, whence $t \in \{4, 5, 6, 7\}$. If $t = 4$ then $r = 3$, $\mu = 1$ and $G$ is 3-regular, contradicting Theorem 2.4. If $t = 5$ or 6 then $\mu = 0$ or $-1$, contrary to assumption. If $t = 7$ then $a = 5$, $\mu = -2$, $H = K_7$ and we obtain a contradiction from Lemma 2.2.

If $(e^*, f^*) = (1, 0)$ then each vertex of $X$ has $2t-4$ neighbours in $X$, and so $r = 2t-2$, $\mu = -2$. Moreover, $H = K_t$ and Lemma 2.2 yields

$$(t-2)^2 - (t+1)(t-2) + 2(t+1) = 0.$$

It follows that $t = 8$. Since $G$ is determined by $H$ and all 6-subsets of $\overline{X}$ as $H$-neighbourhoods of vertices in $X$, we conclude that $G = G_{36}$.

This completes the proof of Theorem 1.1

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