

Rowlinson P (2012) Regular star complements in strongly regular graphs, *Linear Algebra and Its Applications*, 436 (5), pp. 1482-1488.

**This is the peer reviewed version of this article**

*NOTICE: this is the author's version of a work that was accepted for publication in Linear Algebra and Its Applications. Changes resulting from the publishing process, such as peer review, editing, corrections, structural formatting, and other quality control mechanisms may not be reflected in this document. Changes may have been made to this work since it was submitted for publication. A definitive version was subsequently published in Linear Algebra and Its Applications, [VOL 436, ISS 5 (2012)] DOI: <http://dx.doi.org/10.1016/j.laa.2011.08.020>*

# REGULAR STAR COMPLEMENTS IN STRONGLY REGULAR GRAPHS

Peter Rowlinson<sup>1</sup>

Mathematics and Statistics Group  
Department of Computing Science and Mathematics  
University of Stirling  
Scotland FK9 4LA

## Abstract

We prove that, aside from the complete multipartite graphs and graphs of Steiner type, there are only finitely many connected strongly regular graphs with a regular star complement of prescribed degree  $s \in \mathbb{N}$ . We investigate the possible parameters when  $s \leq 5$ .

*AMS Classification:* 05C50

*Keywords:* strongly regular graph, eigenvalue, star complement.

---

<sup>1</sup>Tel.: +44 1786 467468; fax +44 1786 464551; email: p.rowlinson@stirling.ac.uk

# 1 Introduction

Let  $G$  be a finite simple graph of order  $n$  with  $\mu$  as an eigenvalue of multiplicity  $k$ . (Thus the corresponding eigenspace  $\mathcal{E}(\mu)$  of a  $(0, 1)$ -adjacency matrix of  $G$  has dimension  $k$ .) A *star set* for  $\mu$  in  $G$  is a subset  $X$  of the vertex-set  $V(G)$  such that  $|X| = k$  and the induced subgraph  $G - X$  does not have  $\mu$  as an eigenvalue. In this situation,  $G - X$  is called a *star complement* for  $\mu$  in  $G$ . The fundamental properties of star sets and star complements are established in [8, Chapter 5]. A survey of star complements in regular graphs may be found in [18], along with a description of the regular graphs with a star or windmill as a star complement. The cubic graphs with a regular star complement are determined in [15], and the regular graphs with a 1-regular star complement are determined in [17]. As the following examples show, it can happen that a strongly regular graph has a regular star complement. We use the notation of [8].

- Examples 1.1** (i) The Petersen graph has  $3K_2$  as a 1-regular star complement for the eigenvalue  $-2$ .  
(ii) The Petersen graph has  $C_5$  as a 2-regular star complement for the eigenvalue 1.  
(iii) The Gewirtz graph [10] has the Sylvester graph [2, p.223] as a 5-regular star complement for  $-4$ . (The Gewirtz graph has spectrum  $10, 2^{(35)}, -4^{(20)}$ , and the Sylvester graph has spectrum  $5, 2^{(16)}, -1^{(10)}, -3^{(9)}$ .)  
(iv) The complete multipartite graph  $\overline{(s+1)K_u}$  ( $u \in \mathbb{N}$ ) has  $K_{s+1}$  as an  $s$ -regular star complement for the eigenvalue 0.  
(v) The line graph  $L(K_u)$  ( $u > 4$ ) has a union of disjoint odd cycles, of order  $u$ , as a 2-regular star complement for the eigenvalue  $-2$ .

We say that a strongly regular graph is of *Steiner type*  $S(2, \tilde{k}, \tilde{v})$  if its parameters  $n, r, e, f$  coincide with those of the block graph of a Steiner system  $S(2, \tilde{k}, \tilde{v})$ , that is (see [11, Section 9]),

$$n = \frac{\tilde{v}(\tilde{v}-1)}{\tilde{k}(\tilde{k}-1)}, \quad r = \tilde{k} \frac{\tilde{v}-\tilde{k}}{\tilde{k}-1}, \quad e = (\tilde{k}-1)^2 + \frac{\tilde{v}-1}{\tilde{k}-1} - 2, \quad f = \tilde{k}^2. \quad (1)$$

Recall that strongly regular graphs have the same parameters if and only if they are cospectral [8, Section 3.6]. For example, the Chang graphs [8, Example 1.2.6] are of Steiner type because they are cospectral with  $L(K_8)$ , while  $L(K_q)$  is the block graph of the unique design  $S(2, 2, q)$ . We show in Section 2 that, aside from the complete multipartite graphs and graphs of Steiner type, there are only finitely many connected strongly regular graphs with a regular star complement of prescribed degree  $s \in \mathbb{N}$ . Note that complete graphs are excluded from our considerations, and so the case  $s = 0$  does not arise (see Proposition 1.6). In Section 3, we investigate the cases  $s = 1, 2, 3, 4, 5$ . The results are of potential interest in relation to the construction of strongly regular graphs from star complements (cf. Examples 1.3). For instance, the existence of a strongly regular graph with parameters  $(85, 14, 3, 2)$  remains open, but the parameters are consistent with the presence of a 4-regular graph of order 35 as a star complement for  $-3$ .

Here we first recall the required properties of star complements. For  $X \subseteq V(G)$ , we write  $G_X$  for the subgraph of  $G$  induced by  $X$ , and ' $u \sim v$ ' to mean that vertices  $u$  and  $v$  are adjacent.

**Theorem 1.2** [8, Theorem 5.1.7] *Let  $X$  be a set of  $k$  vertices in the graph  $G$  and suppose that  $G$  has adjacency matrix  $\begin{pmatrix} A_X & B^\top \\ B & C \end{pmatrix}$ , where  $A_X$  is the adjacency matrix of  $G_X$ . Then  $X$  is a star set for  $\mu$  in  $G$  if and only if  $\mu$  is not an eigenvalue of  $C$  and*

$$\mu I - A_X = B^\top (\mu I - C)^{-1} B. \quad (2)$$

*In this situation,  $\mathcal{E}(\mu)$  consists of the vectors  $\begin{pmatrix} \mathbf{x} \\ (\mu I - C)^{-1} B \mathbf{x} \end{pmatrix}$  ( $\mathbf{x} \in \mathbb{R}^k$ ).*

Writing  $H = G - X$ , we see that the columns  $\mathbf{b}_u$  ( $u \in X$ ) of  $B$  are the characteristic vectors of the  $H$ -neighbourhoods  $\Delta_H(u) = \{v \in V(H) : u \sim v\}$  ( $u \in X$ ). Thus  $G$  is determined by  $\mu$ , a star complement  $H$  for  $\mu$ , and the  $H$ -neighbourhoods  $\Delta_H(u)$  ( $u \in X$ ).

**Examples 1.3** (i) The Petersen graph can be constructed from a 5-cycle as a star complement  $H$  for 1 by adding 5 vertices whose  $H$ -neighbourhoods are the singleton subsets of  $V(H)$ . It follows from (2) that if  $u, v$  are added, with neighbours  $u', v' \in V(H)$ , then  $u \sim v$  if and only if  $u' \not\sim v'$  [8, Example 5.2.3].

(ii) For odd  $n \geq 5$ , the line graph  $L(K_n)$  can be constructed from an  $n$ -cycle as a star complement  $H$  for  $-2$  by adding  $\frac{1}{2}n(n-3)$  vertices whose  $H$ -neighbourhoods have the form  $\{u_1, u_2, u_3, u_4\}$  with  $u_1 \sim u_2$  and  $u_3 \sim u_4$ . It follows from (2) that if  $u, v$  are added, then  $u \sim v$  if and only if  $\Delta_H(u), \Delta_H(v)$  intersect in two adjacent vertices of  $H$  (cf. [1, Theorem 2.4]).

If  $G$  is  $r$ -regular and  $\mu \neq r$  then the all-1 vector  $\mathbf{j}_n$  is orthogonal to  $\mathcal{E}(\mu)$ ; in other words,  $\mu$  is a non-main eigenvalue (see [16], for example). From the description of  $\mathcal{E}(\mu)$  in Theorem 1.1, we have the following result, where we write  $\mathbf{j}$  for  $\mathbf{j}_{n-k}$ .

**Proposition 1.4** [7, Proposition 0.3] *With the notation above,  $\mu$  is a non-main eigenvalue if and only if*

$$\mathbf{b}_u^\top (\mu I - C)^{-1} \mathbf{j} = -1 \quad \text{for all } u \in X. \quad (3)$$

**Proposition 1.5** *Let  $G$  be an  $r$ -regular graph with an  $s$ -regular subgraph  $H = G - X$  as a star complement for the eigenvalue  $\mu \neq r$ . If  $\mu$  has multiplicity  $k$  then  $|\Delta_H(u)| = s - \mu$  for all  $u \in X$  and*

$$k(r - \mu) = n(r - s). \quad (4)$$

*Proof.* By Proposition 1.4, we have  $-1 = \mathbf{b}_u^\top (\mu I - C)^{-1} \mathbf{j}$ , whence  $\mathbf{b}_u^\top \mathbf{j} = s - \mu$  for each  $u \in X$ . Counting edges between  $X$  and its complement  $\bar{X}$ , we see that  $k(s - \mu) = (n - k)(r - s)$ , equivalently  $k(r - \mu) = n(r - s)$ .  $\square$

It follows that, in the situation of Proposition 1.5,  $\mu$  is an integer, while  $X$  and  $\bar{X}$  form an equitable bipartition of  $V(G)$ ; equivalently,  $X$  and  $\bar{X}$  are regular sets in the sense of [5, 13]. The following observation disposes of the case  $s = 0$ .

**Proposition 1.6** *If  $G$  is an  $r$ -regular graph ( $r > 0$ ) with  $\bar{K}_t$  as a star complement for the eigenvalue  $\mu$  then either*

- (a)  $\mu = -1$  and  $G = tK_{r+1}$ , or
- (b)  $\mu = 1$  and  $G = tK_2$ .

*Proof.* Let  $X$  be a star set for  $\mu$ , with  $H = G - X = K_t$ . Suppose first that  $\mu \neq r$ . Then from Equation (3) we have  $\mathbf{b}_u^\top \mathbf{j} = -\mu$  for each  $u \in X$ . On the other hand, Equation (2) yields  $\mathbf{b}_u^\top \mathbf{b}_u = \mu^2$ , and so  $\mu^2 = -\mu$ . Since  $\mu$  is not an eigenvalue of  $\bar{K}_t$ , we have  $\mu = -1$ ; moreover, each neighbourhood  $\Delta_H(u)$  ( $u \in X$ ) is a singleton. For distinct vertices  $u, v$  in  $X$ , we see from Equation (2) that  $u \sim v$  if and only if  $\mathbf{b}_u^\top \mathbf{b}_v = 1$ , equivalently  $\Delta_H(u) = \Delta_H(v)$ . It follows that each component of  $G$  is complete, and we have case (a).

If  $\mu = r$  let  $v \in X$ , and let  $C$  be the component  $C$  of  $G$  containing  $v$ . Then  $C - v$  is a star complement for  $\mu$  in  $C$ , necessarily a star  $K_{1,r}$ . Since  $G$  is  $r$ -regular and  $r > 0$ , it follows that  $r = 1$ ,  $\mu = 1$ , and we have case (b).  $\square$

## 2 Arithmetic

Let  $G$  be a connected strongly regular graph with parameters  $n, r, e, f$  ( $2 \leq r \leq n - 2$ ). In particular (see [4, 3] for example),

$$(n - r - 1)f = r(r - e - 1) \quad (5)$$

and

$$2(r + 1) \leq n + f. \quad (6)$$

Suppose that  $G$  has an  $s$ -regular star complement  $H = G - X$  for the eigenvalue  $\mu$ , where  $\mu$  has multiplicity  $k$ . Note that  $k \neq 1$  (since  $G$  is not complete) and so  $\mu \neq r$ . Hence  $\mu$  is a non-main eigenvalue. Since  $\mu$  is an integer,  $G$  is not a 5-cycle, and so  $r \geq 3$ . Moreover (see [4, Chapter 2]),

$$\mu = \frac{1}{2}(e - f + \Delta), \quad k = \frac{1}{2}\{n - 1 + \frac{(n-1)(f-e)-2r}{\Delta}\}, \quad (7)$$

where

$$\Delta^2 = (e - f)^2 + 4(r - f). \quad (8)$$

We do not specify a sign for  $\Delta$ . Substituting for  $\mu$ ,  $k$  and  $n$  in Equation (4), we obtain:

$$(2s - r)\Delta = r(e - f + 2). \quad (9)$$

Taking squares and using Equation (8) again, we obtain

$$r^2(r - e - 1) = s(r - s)\Delta^2. \quad (10)$$

Now let  $m$  be the greatest common divisor of  $r$  and  $s$ , say  $r = pm$  and  $s = qm$ , where  $p$  and  $q$  are coprime. Then  $p^2(r - e - 1) = q(p - q)\Delta^2$ , whence  $p^2$  divides  $\Delta^2$ , say  $\Delta^2 = a^2p^2$ , where  $a > 0$ .

**Lemma 2.1** *If  $f < r$  then  $a \leq m$ , with strict inequality when  $s > 2$ .*

*Proof.* Since  $0 < f < r$ , we have  $\Delta^2 \leq (r - 1)^2 + 4(r - 1) < (r + 1)^2$ , whence  $a^2p^2 \leq r^2$  and  $a \leq m$ . If  $a = m$  then Equation (10) becomes  $r - e - 1 = s(r - s)$ , whence  $e = (1 - s)(r - 1 - s)$ . In this situation, if  $s > 1$  then  $s = r - 1$ ,  $e = 0$  and Equation (8) becomes  $(r - f)(r + f - 4) = 0$ . From this it follows that  $r + f = 4$ , and hence that  $r = 3$ ,  $f = 1$ ,  $s = 2$  (cf. Example 1.1(ii)).  $\square$

We are now in a position to prove our finiteness result:

**Theorem 2.2** *For each  $s \in \mathbb{N}$ , there exists a finite family  $\mathcal{R}_s$  of strongly regular graphs with the following property. If  $G$  is a connected strongly regular graph with an  $s$ -regular star complement for the eigenvalue  $\mu$  then exactly one of the following holds:*

- (a)  $\mu = 0$  and  $G = \overline{(s + 1)K_q}$  ( $q \in \mathbb{N}$ ),
- (b)  $\mu = -1 - v$ ,  $s = v(v + 1)$  and  $G$  is of Steiner type  $S(2, v + 1, vw + 1)$  ( $v, w \in \mathbb{N}$ ),
- (c)  $G \in \mathcal{R}_s$ .

*Proof.* If  $f = r$  then  $G$  is a complete multipartite graph (with parts of size  $n - r$ ), say  $G = \overline{(s + 1)K_q}$  ( $q \in \mathbb{N}$ ), with spectrum  $qs, 0^{((q-1)(s+1))}, -s^{(q)}$ . If  $r \neq 2s$  then by Equations (7) and (9), there is a unique solution for  $\mu$ , necessarily  $\mu = 0$ . If  $r = 2s$  then  $q = 2$ , and to verify (a) we must eliminate the possibility  $\mu = -2$ . In this case, a star complement for  $-2$  has order  $s + 2$ , and so is a cocktail party graph (of order at least 4): this is a contradiction because such a graph has  $-2$  as an eigenvalue. Thus (a) holds when  $f = r$ , and we now assume that  $f < r$ .

We write  $\alpha = \frac{a}{m}$ , so that Equation (10) becomes

$$e = s^2\alpha^2 - 1 - r(s\alpha^2 - 1). \quad (11)$$

Substituting for  $e$  and  $\Delta^2$  in Equation (8), and solving the resulting quadratic in  $f$ , we obtain:

$$f = s^2\alpha^2 + 1 - r(s\alpha^2 - 1) \pm \alpha(r - 2s). \quad (12)$$

We have

$$\frac{r(r - e - 1)}{f} = \frac{rs(r - s)\alpha^2}{s^2\alpha^2 + 1 - r(s\alpha^2 - 1) \pm \alpha(r - 2s)},$$

and this is an integer by Equation (5). It is expressible as the quotient  $(Jp^2 + Kp)/(Lp + M)$ , where

$$J = qma^2, \quad K = -q^2ma^2, \quad L = m - qa^2 \pm a, \quad M = q^2a^2 + 1 \mp 2aq.$$

Now  $L^2(Jp^2 + Kp) = J(Lp + M)^2 + (Lp + M)(LK - 2JM) + M(JM - KL)$  and so  $Lp + M$  divides  $M(JM - KL)$ . This enables us to bound  $Lp + M$  when  $M(JM - KL) \neq 0$ .

Consider the first choice of sign, and note that the argument in this case embraces the case  $r = 2s$ . We have  $M \neq 0$  for otherwise  $qa = 1$  and then  $a = 1, s = m, \alpha = \frac{1}{s}$ , whence  $f = r$ , contrary to assumption. Also  $JM - KL = qma^2(mq - qa + 1)$ , and this is non-zero by Lemma 2.1. Consequently  $M(JM - KL) \neq 0$ . From Lemma 2.1 we see also that, for given  $s$ , there are only finitely many possibilities  $m, a, q$ . If  $L = 0$  then  $m = a(qa - 1)$  and Equation (11) yields

$$s^2\alpha^2 - e - 1 = r(s\alpha^2 - 1) = \frac{ra}{m}, \quad (13)$$

whence  $r < s^2a/m$ . Then there are only finitely many possibilities for  $r$ , and (since  $n \leq r^2 + 1$ ) only finitely many possibilities for  $n$ . On the other hand, if  $L \neq 0$  then the relation  $|Lp + M| \leq |M(JM - KL)|$  shows that  $p$  (and hence  $r$  and hence  $n$ ) is bounded in terms of  $m, a, q$  (and hence in terms of  $s$ ).

Now consider the second choice of sign, with  $r \neq 2s$ . Here  $M = (qa + 1)^2 \neq 0$ , and  $JM - KL = qma^2(1 + qa + qm) \neq 0$ . Thus  $M(JM - KL) \neq 0$ , and if  $L \neq 0$  then  $n$  is bounded as before. If  $L = 0$  then  $m = a(qa + 1)$  and so  $r = pa(qa + 1)$ . From Equations (11) and (12), we have  $e = q^2a^2 + pa - 1$ ,  $f = (qa + 1)^2$ . It follows that  $n = (pa + 1)(pqa^2 + qa + 1)/(qa + 1)$ . Comparing these parameters with those in (1), we see that  $G$  is of Steiner type  $S(2, v + 1, vw + 1)$ , where  $v = qa$  and  $w = pa + 1$ . In this situation, Equation (9) yields  $\Delta = r(e - f + 2)/(2s - r) = -qa$  and so  $\mu = \frac{1}{2}(e - f + \Delta) = -1 - qa = -1 - v$ . Finally,  $s = qa(qa + 1) = v(v + 1)$ , and so we have case (b) of the Theorem.  $\square$

Note that  $\mathcal{R}_1$  contains the Petersen graph, and in view of [17, Proposition 3.1],  $\mathcal{R}_s$  contains the graph  $\overline{L(K_{s+3})}$  whenever  $s > 1$ : if  $H$  is a 2-regular star complement for  $-2$  in  $L(K_{s+3})$  then  $\overline{H}$  is an  $s$ -regular star complement for 1 in  $L(K_{s+3})$ . Example 1.1(iii) shows that  $\mathcal{R}_5$  contains the Gewirtz graph, and that a strongly regular graph with a regular star complement is not necessarily of the form  $L(K_q)$  or  $\overline{L(K_q)}$ .

### 3 Regular star complements of small degree

In this section we investigate the parameters of  $G$  that arise when  $s \leq 5$ . We retain the notation of Section 2, and exclude the complete multipartite graphs by taking  $f < r$ . The possibilities for the parameters of a graph in  $\mathcal{R}_s$  ( $s \leq 5$ ) are listed in the accompanying table; information on the existence and uniqueness of the corresponding graphs may be found at <http://www.win.tue.nl/~aeb/graphs/srg/srgtab.html>, courtesy of A. E. Brouwer. There are at least 32649 strongly regular graphs whose parameters appear in the table, and the graphs themselves are not investigated here.

|       |    |    |    |    |    |    |    |    |    |    |     |    |
|-------|----|----|----|----|----|----|----|----|----|----|-----|----|
| $s$   | 1  | 2  | 3  | 4  | 4  | 4  | 4  | 4  | 5  | 5  | 5   | 5  |
| $\mu$ | -2 | 1  | 1  | 1  | 1  | -3 | -3 | -3 | 1  | -3 | -3  | -4 |
| $n$   | 10 | 10 | 15 | 10 | 21 | 26 | 45 | 85 | 28 | 36 | 126 | 56 |
| $r$   | 3  | 3  | 6  | 6  | 10 | 10 | 12 | 14 | 15 | 15 | 25  | 10 |
| $e$   | 0  | 0  | 1  | 3  | 3  | 3  | 3  | 3  | 6  | 6  | 8   | 0  |
| $f$   | 1  | 1  | 3  | 4  | 6  | 4  | 3  | 2  | 10 | 6  | 4   | 2  |

With minor variations, the parameters are found as follows. For prescribed  $s, a, m$  we use Equation (10) to find  $e$  in terms of  $r$ , and Equation (8) to determine the (one or two) possibilities for  $f$  before imposing the condition that  $f$  divides  $r(r-e-1)$ . When  $r \neq 2s$  we find  $\Delta$  from Equation (9),  $\mu$  from Equation (7). We give just an outline of the calculations, together with a description of the graphs involved where appropriate.

*The case  $s = 1$ .* Here  $r^2(r-e-1) = (r-1)\Delta^2$ , whence  $\Delta^2 = r^2$ ,  $e = 0$  and Equation (8) becomes  $(r-f)(r+f-4) = 0$ . Thus  $r+f = 4$  and so  $(n, r, e, f) = (10, 3, 0, 1)$ . From Equation (9), we have  $\Delta = -3$  and so  $\mu = -2$ . In this case,  $G$  is the Petersen graph, arising as in Example 1.1(i). This result is just a special case of [17, Theorem 3.2].

*The case  $s = 2$ .* Here  $r^2(r-e-1) = 2(r-2)\Delta^2$ . If  $r$  is odd then  $r^2 = \Delta^2$  and  $r+e = 3$ . Hence  $r = 3$ ,  $e = 0$  and we find in turn that  $f = 1$ ,  $n = 10$ ,  $\Delta = 3$ ,  $\mu = 1$ . Thus  $G$  is the Petersen graph, arising as in Example 1.1(ii).

If  $r$  is even then  $r^2 \neq \Delta^2$  (by the argument above) and so  $r^2 = 4\Delta^2$  by Lemma 2.1. It follows that  $r = 2e$ , and then Equation (8) becomes  $(f-4)(f-2e) = 0$ . Hence  $f = 4$ ,  $e \neq 2$  and  $n = \frac{1}{2}(e+1)(e+2)$ . From Equation (9), we have  $\Delta = -e$ , and so  $\mu = -2$ . The parameters of  $G$  are those of  $L(K_u)$ , where  $u = e+2$ , and so  $G$  is cospectral with  $L(K_u)$ . Thus either  $G = L(K_u)$  or  $u = 8$  and  $G$  is a Chang graph (see [7, Chapter 4]). All three Chang graphs arise in case (b) of Theorem 2.2 because each has  $C_3 \dot{\cup} C_5$  as a star complement for  $-2$ .

*The case  $s = 3$ .* Here  $r^2(r-e-1) = 3(r-3)\Delta^2$ , and by Lemma 2.1, either  $\Delta^2 = \frac{1}{9}r^2$  or  $\Delta^2 = \frac{4}{9}r^2$ .

If  $\Delta^2 = \frac{1}{9}r^2$  then  $e = \frac{2}{3}r$ ,  $f = \frac{1}{3}r + 4$  and

$$n - r - 1 = \frac{r(r-e-1)}{f} = \frac{r(r-3)}{r+12}.$$

It follows that  $r+12$  divides 180. Since  $f \neq r > 3$  and  $r$  is divisible by 3, we have  $r \in \{18, 24, 33, 48, 78, 168\}$ . The cases  $r = 24, 48, 168$  do not arise because they lead to non-integer values of  $k$  in (7). The cases  $r = 18, 33, 78$  are ruled out by the ‘absolute bound’:  $n \leq \frac{1}{2}k'(k'+3)$ , where  $k'$  is the multiplicity of either multiple eigenvalue (see [19, Section 6] or [8, Theorem 3.6.7]).

If  $\Delta^2 = \frac{4}{9}r^2$  then  $\frac{1}{3}r = 3 - e$ . Here the possibilities for  $(n, r, e, f)$  are  $(28, 9, 0, 4)$  and  $(15, 6, 1, 3)$ , with associated spectra  $28, 1^{(21)}, -5^{(6)}$  and  $5, 1^{(9)}, -3^{(5)}$  respectively. The absolute bound is violated in the first case. In the second case, by considering  $\overline{G}$  we see that  $G = \overline{L(K_6)}$ , an example noted in Section 2. Here  $\mu = 1$  because a star complement  $H$  for  $\mu$  has even



order; thus  $H$  is a 3-regular graph of order 6. Since 1 is an eigenvalue of  $\overline{C_6}$ , necessarily  $H = \overline{2K_3}$ .

*The case  $s = 4$ .* Here  $r^2(r - e - 1) = 4(r - 4)\Delta^2$ , and by Lemma 2.1,  $\Delta^2 \in \{\frac{1}{16}r^2, \frac{1}{4}r^2, \frac{9}{16}r^2\}$ .

If  $\Delta^2 = \frac{1}{16}r^2$  then  $e = \frac{3}{4}r$  and Equation (8) yields  $f = \frac{1}{2} + 4$ . Writing  $r = 4p$ , we see that

$$\frac{r(r - e - 1)}{f} = \frac{2p(p - 1)}{p + 2},$$

whence  $p + 2$  divides 12. Since  $f < r$ , we deduce that either  $(n, r, e, f) = (21, 16, 12, 12)$  or  $(n, r, e, f) = (56, 40, 30, 24)$ . In both cases, condition (6) is violated.

If  $\Delta^2 = \frac{1}{4}r^2$  then  $e = 3$  and  $f \in \{\frac{1}{2}r + 1, 9 - \frac{1}{2}r\}$ . If  $f = \frac{1}{2}r + 1$  then  $(n, r, e, f) \in \{(10, 6, 3, 4), (21, 10, 3, 6), (56, 22, 3, 12)\}$ . Here the third possibility is ruled out by the absolute bound. If  $(n, r, e, f) = (10, 6, 3, 4)$  then  $G = L(K_5)$ ,  $\mu = 1$  and  $H = \overline{3K_2}$ , an example complementary to Example 1.1(i). If  $(n, r, e, f) = (21, 10, 3, 6)$  then  $G = L(K_7)$ . Secondly, suppose that  $f = 9 - \frac{1}{2}r$ . Then  $(n, r, e, f) \in \{(17, 8, 3, 5), (26, 10, 3, 4), (45, 12, 3, 3), (85, 14, 3, 2), (209, 16, 3, 1)\}$ . The first possibility is excluded by the requirement that  $f \mid r(r - e - 1)$ , and the last by the condition: if  $f = 1$  then  $r \geq (e+1)(e+5)$  [9, Theorem 4.2]. The graphs with  $(n, r, e, f) = (26, 10, 3, 4)$  are complements of graphs of Steiner type  $S(2, 3, 13)$ , and there are 10 of them (see [14]). There are 78 graphs with  $(n, r, e, f) = (45, 12, 3, 3)$  [6]. The existence of a strongly regular graph with parameters  $(85, 14, 3, 2)$  remains an open question.

If  $\Delta^2 = \frac{9}{16}r^2$  then  $5r = 32 - 4e$ , impossible since  $r \geq 5$ .

*The case  $s=5$ .* Here  $r^2(r - e - 1) = 5(r - 5)\Delta^2$ , and by Lemma 2.1,  $r = 5p$  for some integer  $p > 1$ . Moreover,  $\Delta^2 \in \{\frac{1}{25}r^2, \frac{4}{25}r^2, \frac{9}{25}r^2, \frac{16}{25}r^2\}$ .

If  $\Delta^2 = \frac{1}{25}r^2$  then  $e = \frac{4}{5}r$  and  $f = \frac{3}{5}r + 4$ . Since  $r(r - e - 1)/f = 5p(p - 1)/3p + 4$  and  $f < r$ , we have  $p \in \{8, 22\}$ . Then  $(n, r, e, f) = (51, 40, 32, 28)$  or  $(144, 110, 88, 70)$ , and in both cases condition (6) is violated.

If  $\Delta^2 = \frac{4}{25}r^2$  then  $r = 5(e - 3)$  and  $f \in \{3e - 8, 12 - e\}$ . We find that  $(n, r, e, f) = (28, 15, 6, 10)$  or  $(144, 65, 16, 40)$ . In the first case,  $\mu = 1$  and  $\overline{G}$  is either  $L(K_8)$  or a Chang graph. The second case is ruled out by the absolute bound. When  $f = 12 - e$  we find that  $r = 5p$ , where  $1 < p < 9$  and  $9 - p$  divides  $5p(4p - 3)$ . Hence  $p \in \{3, 5\}$  and  $(n, r, e, f) = (36, 15, 6, 6)$  or  $(126, 25, 8, 4)$ . In both cases,  $\mu = -3$ . There are 32548 strongly regular graphs with parameters  $(36, 15, 6, 60)$  [12], while a strongly regular graph with parameters  $(126, 25, 8, 4)$  is described in [3].

If  $\Delta^2 = \frac{9}{25}r^2$  then  $r = 5p$ ,  $e = 8 - 4p$  and necessarily  $r = 10, e = 0$ . Then  $(n, r, e, f) = (56, 10, 0, 2)$  and  $G$  is the Gewirtz graph [10]. Here  $\mu = -4$  because a 5-regular star complement has even order (cf. Example 1.1(iii)).

If  $\Delta^2 = \frac{16}{25}r^2$  then  $r = 5p$  and  $e = -11p + 15$ , impossible since  $p > 1$ .

## References

- [1] F. K. Bell, Characterizing line graphs by star complements, *Linear Algebra Appl.* 296 (1999), 15-25.
- [2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag (Berlin), 1989.
- [3] A. E. Brouwer and J. H. van Lint, Strongly regular graphs and partial geometries, in: *Enumeration and Design*, Proc. Silver Jubilee Conf. on Combinatorics, Waterloo, 1982 (eds. D. M. Jackson and S. A. Vanstone), Academic Press (Toronto) 1984, pp.85-122.
- [4] P. J. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, Cambridge University Press (Cambridge), 1991.
- [5] D. M. Cardoso and P. Rama, Equitable bipartitions of graphs and related results, *J. Math. Sci. (N. Y.)* 120 (2004), 869-880.
- [6] K. Coolsaet, J. Degraer and E. Spence, The strongly regular  $(45, 12, 3, 3)$  graphs, *Electr. J. Combin.* **13** (2006) R32.
- [7] D. Cvetković, P. Rowlinson and S. K. Simić, *Spectral Generalizations of Line Graphs*, Cambridge University Press (Cambridge), 2004.
- [8] D. Cvetković, P. Rowlinson and S. K. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press (Cambridge), 2009.
- [9] J. Deutsch and P. H. Fisher, On strongly regular graphs with  $\mu = 1$ , *Europ. J. Combinatorics* **22** (2001), 303-306.
- [10] A. Gewirtz, The uniqueness of  $g(2, 2, 10, 56)$ , *Trans. New York Acad. Sci.* **31** (1969), 656-675.
- [11] X. L. Hubaut, Strongly regular graphs, *Discrete Math.* **13** (1975), 357-381.
- [12] B. D. McKay and E. Spence, Classification of regular two-graphs on 36 and 38 vertices, *Australasian J. Combin.* **24** (2001), 293-300.
- [13] A. Neumaier, Regular sets and quasi-symmetric 2-designs, *Lecture Notes in Math.* 969, Springer-Verlag (Berlin), 1982, pp.258-275.
- [14] A. J. L. Paulus, *Conference Matrices and Graphs of Order 26*, Technological University of Eindhoven, T. H. Report 73-WSK-06, 1973.
- [15] P. Rowlinson, Star partitions and regularity in graphs, *Linear Algebra Appl.* 226-228 (1995), 247-265.

- [16] P. Rowlinson, The main eigenvalues of a graph: a survey, *Appl. Anal. Discrete Math.* 1 (2007), 455-471.
- [17] P. Rowlinson, On induced matchings as star complements in regular graphs, *J. Math. Sciences*, to appear.
- [18] P. Rowlinson and B. Tayfeh-Rezaie, Star complements in regular graphs: old and new results, *Linear Algebra Appl.* 432 (2010), 2230-2242.
- [19] J. J. Seidel, Strongly regular graphs, in: *Surveys in Combinatorics* (ed. B. Bollobàs), Cambridge University Press (1979), pp.157-180.