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Co-cliques and star complements in extremal strongly regular graphs

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Abstract

Suppose that the positive integer μ is the eigenvalue of largest multiplicity in an extremal strongly regular graph G. By interlacing, the independence number of G is at most $4\mu^2 + 4\mu - 2$. Star complements are used to show that if this bound is attained then either (a) $\mu = 1$ and G is the Schläfli graph or (b) $\mu = 2$ and G is the McLaughlin graph.

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1. Introduction

Let G be a simple graph of order n with an eigenvalue μ of multiplicity k, and let $t = n - k$; thus the eigenspace $\mathscr{E}(\mu)$ of a (0, 1)-adjacency matrix of G has dimension k and codimension t.

We recall some terminology and results from [4, Chapter 7]. We write $V(G)$ for the vertex-set of G, and use the notation '$u \sim v$' to indicate that vertices u, v are adjacent. A *star set* for μ in G is a subset X of $V(G)$ such that $|X| = k$ and μ is not an eigenvalue of $G - X$. The induced subgraph $G - X$ is called a *star complement* for μ in G (or, in [6], a μ-*basic subgraph* of G). Star sets and star complements exist for any eigenvalue of any graph, and the means of constructing all graphs with a prescribed star complement are described in [5, Chapter 5].

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Examples 1.1. (i) The Schläfli graph Sch_{10} [3, p. 22] has eigenvalues 10, 1, −5 of multiplicities 1, 20, 6, respectively. Star complements for the eigenvalue 1 include induced subgraphs $K_{1,2} \dot{\cup} 4K_1$ and $K_{1,5} \dot{\cup} K_1$.

(ii) The McLaughlin graph McL_{112} [8] has eigenvalues 112, 2, −28 of multiplicities 1, 252, 22, respectively. It is the graph of largest order with $K_{1,16} \dot{\cup} 6K_1$ as a star complement for the eigenvalue 2 [9].

In [1], star complements were used to show that if G is regular, if $t > 2$ and if $\mu \neq -1$ or 0 then $n \leqslant \frac{1}{2}(t-1)(t+2)$, with equality if and only if G is an extremal strongly regular graph, defined as follows.

A graph G is said to be strongly regular with parameters n, r, e, f if it is r-regular of order n, any two adjacent vertices have e common neighbours and any two non-adjacent vertices have f common neighbours. To exclude degenerate cases, we suppose that $f > 0$ and $r < n-1$. Then G has precisely 3 distinct eigenvalues, say λ, μ and r of multiplicities m, k, 1, respectively, where $1 < m \leqslant k$. We have the 'absolute bound' $n \leqslant \frac{1}{2}m(m+3)$ [10], and G is said to be extremal if this bound is attained.

A Smith graph is a strongly regular graph whose parameters are rational functions of integer eigenvalues λ and μ: these functions arise from [11] and are specified in [3, p. 111]. In an extremal Smith graph, the parameters are polynomial functions of a single eigenvalue, given explicitly in Section 2. An extremal strongly regular graph is a pentagon, a complete multipartite graph or a Smith graph (see [10, Section 7] and [2, Theorem 6.6]). The only known extremal Smith graphs are those in Examples 1.1 (where $\mu > 0$) together with their complements (for which $\mu < 0$). It is an open question of at least 30 years' standing as to whether these are the only examples. We show that there are no further examples in which an independent set of vertices has the largest possible size, namely $m = t - 1$. Explicitly, we prove that there are no further examples in which the eigenvalue of largest multiplicity is positive and a corresponding star complement has the form $K_{1,s} \dot{\cup} (t-s-1)K_1$ $(2 \leqslant s \leqslant t-1)$. The proof reduces the restricted question to a problem on designs, and then uses the fact there are very few tight 4-designs (see Section 2).

We use the following formulation of [4, Theorems 7.4.1 and 7.4.4].

Theorem 1.2. *Let X be a set of k vertices in the graph G and suppose that G has adjacency matrix* $\begin{pmatrix} A_X & B^{\mathrm{T}} \\ B & C \end{pmatrix}$, *where A_X is the adjacency matrix of the subgraph induced by X. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^{\mathrm{T}}(\mu I - C)^{-1}B.$$

In this situation, the eigenspace of μ consists of the vectors

$$\begin{pmatrix} \mathbf{v} \\ (\mu I - C)^{-1}B\mathbf{v} \end{pmatrix} \quad (\mathbf{v} \in \mathbb{R}^k).$$

It follows that if X is a star set for μ in G, and $H = G - X$, then G is determined by μ, H and the H-neighbourhoods of the vertices in X. We shall use implicitly the fact that if $\mu \neq 1$ or 0 then these neighbourhoods are distinct and non-empty [5, Proposition 5.1.4].

When $V(G) = \{1, \ldots, n\}$ and X is the star set $\{1, \ldots, k\}$, we extend the notation of Theorem 1.2 by defining

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^{\mathrm{T}}(\mu I - C)^{-1}\mathbf{y} \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}^t).$$

We denote the columns of B by $\mathbf{b}_u$ $(u = 1, \ldots, k)$. Now it follows from Theorem 1.2 that, for all vertices u, v of X:

$$\langle \mathbf{b}_u, \mathbf{b}_v \rangle = \begin{cases} \mu & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Moreover, if G is r-regular, if $\mu \neq r$ and if $\mathbf{j}$ denotes the all-1 vector in $\mathbb{R}^t$ then

$$\langle \mathbf{b}_u, \mathbf{j} \rangle = -1 \quad (u = 1, \ldots, k), \tag{2}$$

since the all-1 vector in $\mathbb{R}^n$ is orthogonal to the eigenspace of μ.

2. Extremal Smith graphs

We retain the notation of Section 1, and we take G to be an extremal Smith graph. By passing to the complement of G if necessary we may assume that the eigenvalue μ of G of largest multiplicity is a positive integer. Then the various parameters of G are expressible in terms of μ as follows (see [3, Chapter 8] and [1]):

$$\begin{aligned} n &= (2\mu + 1)^2(2\mu^2 + 2\mu - 1), \\ r &= 2\mu^3(2\mu + 3), \\ e &= \mu(2\mu - 1)(\mu^2 + \mu - 1), \\ f &= \mu^3(2\mu + 3), \\ \lambda &= -\mu^2(2\mu + 3), \\ k &= 2\mu(\mu + 1)(2\mu - 1)(2\mu + 3), \\ t &= 4\mu^2 + 4\mu - 1. \end{aligned} \tag{3}$$

We now suppose that G has a star complement H for μ of the form $K_{1,s} \dot\cup (t - s - 1)K_1$ $(2 \leqslant s \leqslant t - 1)$. Thus the adjacency matrix C of H has minimal polynomial $m(x) = x(x^2 - s)$ and

$$m(\mu)(\mu I - C)^{-1} = (\mu^2 - s)I + \mu C + C^2. \tag{4}$$

It follows that:

$$\mu(\mu^2 - s)\langle \mathbf{j}, \mathbf{j} \rangle = (\mu^2 - s)t + 2s\mu + s + s^2. \tag{5}$$

On the other hand, we know from [1, Eq. (14)] that $\langle \mathbf{j}, \mathbf{j} \rangle = n(\mu - r)^{-1}$, and after expressing t, n, r in terms of μ we obtain

$$s^2 - s(4\mu^2 + 4\mu - 1) + 4\mu^4 + 6\mu^3 = 0, \quad \text{that is,} \quad r = s(t - s). \tag{6}$$

We note in passing that the diophantine equation (6) has infinitely many solutions when $4\mu^2 + 6\mu - 1$ is 3 times a square, as for example when $\mu = 14$ and $s = 343$. Accordingly we look to more sophisticated means of proving that $\mu = 1$ or 2. First, it turns out that the initial steps of the construction of McL_{112} from the star complement $K_{1,16} \dot\cup 6K_1$ (cf. [3,7,9]) can be replicated in the general case, as we now describe.

Let $V(H) = R \dot\cup S \dot\cup T$, where $|R| = 1$, $|S| = s$, $|T| = t - s - 1$ and the vertex in R is adjacent to each vertex of S. Following [7, Chapter 7], we say that a vertex u in X is of type (a, b, c) if its H-neighbourhood consists of a vertices in R, b vertices in S and c vertices in T. Now $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$, and if we use (4) to compute $\langle \mathbf{b}_u, \mathbf{b}_u \rangle$, we obtain the following (cf. [7, Eq. (7.2)]):

$$\mu^2(\mu^2 - s) = (\mu^2 - s)(a + b + c) + 2\mu ab + a^2 s + b^2. \tag{7}$$

From (2), we have

$$-\mu(\mu^2-s)=(\mu^2-s)(a+b+c)+\mu(as+b)+(a+b)s. \tag{8}$$

On subtracting (8) from (7), and noting that $a^2=a$, we obtain

$$b^2-b(s+\mu-2\mu a)+(s-\mu^2)(\mu^2+\mu)-\mu as=0. \tag{9}$$

When $a=0$ the solutions of (9), as a quadratic in b, are $b=s-\mu^2, b=\mu^2+\mu$; and when $a=1$ the solutions are $b=s-\mu^2-\mu, b=\mu^2$. It follows from (7) that there are at most four types of vertex in X: let $X=U\dot{\cup}V$ where U consists of the vertices of type $(0,\mu^2+\mu,t-s-\mu^2-\mu)$ or $(1,\mu^2,t-s-\mu^2)$ and V consists of the vertices of type $(0,s-\mu^2,\mu^2)$ or $(1,s-\mu^2-\mu,\mu+\mu^2)$.

Next, let u, v be distinct vertices in X, of types (a,b,c), (α,β,γ), respectively, with H-neighbourhoods $\Delta(u),\Delta(v)$. Let $\rho_{uv}=|\Delta(u)\cap\Delta(v)|$, and let $a_{uv}=1$ or 0 according as $u\sim v$, $u\nsim v$. From Eqs. (1) and (4) we have (cf. [7, Eq. (7.4)])

$$(\rho_{uv}+\mu a_{uv})(s-\mu^2)=a\alpha s+b\beta+\mu(a\beta+\alpha b). \tag{10}$$

From this we find that

$$|\Delta(u)\cap\Delta(v)\cap(S\cup T)|+\mu a_{uv}=\begin{cases}t-s-\mu^2 & \text{if } u,v\in U,\\ s-\mu^2 & \text{if } u,v\in V,\\ \mu^2+\mu & \text{if } u\in U,\ v\in V.\end{cases} \tag{11}$$

Now let G^* be the graph obtained from G by taking the complement, adding an isolated vertex x and switching with respect to $\{x\}\dot{\cup}U$. Let $R=\{y\}$ and $\{x\}=W$. Thus $V(G^*)=Y\dot{\cup}K$, where $Y=R\dot{\cup}U\dot{\cup}V$ and $K=W\dot{\cup}S\dot{\cup}T$. Note that K induces a clique in G^* of order t. We say that a vertex in Y is of type $[a',b',c']$ in G^* if it is adjacent to a' vertices in W, b' vertices in S and c' vertices in T. Vertices in U are of type $[0,\mu^2+\mu,t-s-\mu^2-\mu]$ or $[0,\mu^2,t-s-\mu^2]$, vertices in V are of type $[1,\mu^2,t-s-1-\mu^2]$ or $[1,\mu^2+\mu,t-s-1-\mu^2-\mu]$, and y is of type $[1,0,t-s-1]$. Let $\Delta^*(w)$ denote the K-neighbourhood of a vertex $w\in Y$. It follows from (11) that for distinct vertices w_1, w_2 in Y:

$$|\Delta^*(w_1)\cap\Delta^*(w_2)|=\begin{cases}t-s-\mu^2 & \text{if } w_1\sim w_2 \text{ in } G^*,\\ t-s-\mu^2-\mu & \text{if } w_1\nsim w_2 \text{ in } G^*.\end{cases} \tag{12}$$

We note that Y contains both adjacent and non-adjacent pairs of vertices of G^*. This follows, for example, by consideration of the Seidel adjacency matrix of G^*, whose eigenvalues are $2\mu+1$, $2\lambda+1$ of multiplicities $k+1$, t, respectively (cf. [3, Theorems 4.9 and 4.11]): by interlacing, the subgraph induced by Y has $2\mu+1$ as a Seidel eigenvalue. Now let $\mathscr{B}=\{\Delta^*(w):w\in Y\}$. The system $(K,\mathscr{B})$ has the properties

(i) $|K|=t$ and $|\mathscr{B}|=\binom{t}{2}$,
(ii) $|B|=t-s$ for each $B\in\mathscr{B}$,
(iii) for distinct B, $B'\in\mathscr{B}$, $|B\cap B'|$ takes exactly two values.

It follows that if $4\leqslant t-s<t-2$ then $(K,\mathscr{B})$ is a tight 4-design in the sense of [3, Chapter 1] and hence necessarily the Steiner system $S(4,7,23)$ (see [3, Theorems 1.52 and 1.54]). In these circumstances, $\mu=2$ and we can reverse the construction of G^* from G to obtain McL_{112}.

It remains to consider the cases $s=2$, $s=t-3$, $s=t-2$ and $s=t-1$. If $s=2$ then from Eq. (6) we obtain $(\mu-1)(2\mu^3+2\mu^2+\mu-3)=0$, whence $\mu=1$, $H=K_{1,2}\dot{\cup}4K_1$ and

$G = \mathrm{Sch}_{10}$. In the remaining cases we exploit Eq. (6) in the form $r = s(t - s)$. If $s = t - 3$ then from the relations (3) we obtain $\mu^3(2\mu + 3) = 6(\mu^2 + \mu - 1)$, impossible by parity considerations. If $s = t - 2$ then we obtain $(\mu - 1)(2\mu^3 + 5\mu^2 + \mu - 3) = 0$, whence $\mu = 1$, $H = K_{1,5} \dot\cup K_1$ and $G = \mathrm{Sch}_{10}$. Finally, if $s = t - 1$ then we obtain the contradiction $(\mu^2 + \mu - 1)(2\mu^2 + \mu - 1) = 0$.

Thus Sch_{10} and McL_{112} are the only extremal Smith graphs satisfying our hypotheses. These are the only extremal strongly regular graphs which arise because the pentagon is excluded by the assumption that μ is an integer, and complete multipartite graphs are excluded because the eigenvalue of largest multiplicity in such graphs is 0. Accordingly we may summarize our results as follows.

Theorem 2.1. *Let G be an extremal strongly regular graph in which an eigenvalue μ of largest multiplicity is a positive integer. If a star complement for μ has the form $K_{1,s} \dot\cup (t - s - 1)K_1$ $(2 \leqslant s \leqslant t - 1)$ then either*

(a) $\mu = 1$, $t = 7$, $s \in \{2, 5\}$ *and G is the Schläfli graph, or*
(b) $\mu = 2$, $t = 23$, $s = 16$ *and G is the McLaughlin graph.*

Corollary 2.2. *Let G be an extremal strongly regular graph in which an eigenvalue μ of largest multiplicity is positive. Then the independence number of G is at most $4\mu^2 + 4\mu - 2$, with equality if and only if G is a pentagon, the Schläfli graph, or the McLaughlin graph.*

Proof. We may suppose that μ is an integer greater than 1, for otherwise G is a pentagon or Sch_{10}. By interlacing, the largest eigenvalue of an induced subgraph of order t is at least μ. It follows that G has no co-clique of order t, and no induced subgraph $K_2 \dot\cup (t - 2)K_1$. Now suppose that G has a co-clique C of order $t - 1$ $(= 4\mu^2 + 4\mu - 2)$. Then each vertex v outside C is adjacent to at least 2 vertices of C; in other words, $C + v$ has the form $K_{1,s} \dot\cup (t - s - 1)K_1$ $(2 \leqslant s \leqslant t - 1)$. The vertex v can be chosen such that $s \neq \mu^2$ for otherwise, counting in two ways the number of edges with a vertex in C, we have $r(t - 1) = \mu^2(n - t + 1)$, whence $4\mu(2\mu + 3) = (2\mu + 1)^2 - 2$, a contradiction. Accordingly v may be chosen such that $C + v$ is a star complement for μ. We now conclude from Theorem 2.1 that $\mu = 2$ and G is the McLaughlin graph. □

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