Abstraction and Identity

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1 Introduction

Over the last 20 years there has been a resurrection of a position in the philosophy of mathematics - abstractionism, or Neo-Fregeanism - which aims to provide a reduction of standard mathematical practice (or at least the epistemology and ontology underlying such practise) in terms of abstraction principles embedded within second-order logic. Recent attempts have shown that arithmetic and analysis can be reduced to abstraction principles of the relevant kind, although the prospects for set theory seem, at present, to be less promising.[1]

Abstraction principles are second-order statements of the form:

$$AP_{@}\text{: } (\forall X)(\forall Y)[@(X) = @(Y) \leftrightarrow E_{@}(X, Y)]$$

where "@" is a term-forming operator taking concepts as argument and providing objects as output and "$E_{@}$" is an equivalence relation on concepts[2].

A quick note on notation: "@" is used to represent arbitrary abstraction operators. "§" is used to represent the extension- (or set-) forming operator, and "♯" will be used to represent numerical abstraction operators. Any object that falls in the range of an abstraction operator "@" is called an abstract, and, given a particular abstraction operator "@", we will call terms of the form "$@(P)$" @-abstraction terms, and the corresponding objects @-abstracts.

Abstraction principles are taken to fix the truth conditions for identity statements regarding abstracts (i.e. the truth conditions of the identity on the left-hand side of the abstraction principle) in virtue of the equivalence

[1] Wright [1983] contains a formal and philosophical reduction of arithmetic to the abstraction principle known as *Hume's Principle*. See Hale [2000] on analysis and Cook [2001] for a critique, and Boolos [1989] and Cook [2003] for attempts to provide Neo-Fregean accounts of set theory. Fine [2002] and MacBride [2000] contain good discussions of various issues surrounding abstractionism.

[2] Here we restrict our attention to conceptual abstraction and ignore objectual abstraction principles such as the pairing or direction principles.

relation on the right-hand side. As a result, abstraction principles provide necessary and sufficient conditions for abstracts of the same kind to be identical. As has often been noted, however, abstraction principles are silent with regard to the truth conditions of mixed identity statements of the form:

$$@(P) = t$$

where "t" is not a term of the form "$@(P)$" for some "P". This is the (in)famous *Caesar Problem*, which Frege himself was aware of. When considering various means for defining number, he points out that:

> ...we can never, to take a crude example, decide by means of our definitions whether any concept has the number JULIUS CAESAR belonging to it, or whether that same familiar conqueror of Gaul is a number or not (Frege [1884], p. 68)

The *Caesar Problem* has received significant attention in recent years.[3]. In this paper, however, we focus on a particular case of the *Caesar Problem* that surprisingly has received little attention on its own.[4] We can isolate the problem by considering the following hypotheses:

1. All abstracts exist necessarily.

2. Only abstracts exist necessarily.

If we were to accept (1) and (2), then we could quickly dispense with the traditional version of the *Caesar Problem* since (presumably) Caesar exists contingently whereas numbers (and other abstracts) do not.

Of course, if the arguments to follow depended on these two theses, then (1) and (2) stand in need of some motivation and defence. We will not propose such a defence here, however (it is worth pointing out that, although (2) seems to us plausible, the existence of sets of contingent objects renders (1) questionable at best[5]). Instead, we will merely point out that even if we were to accept (1) and (2), a version of the *Caesar Problem* remains.

While acceptance of (1) and (2) settles questions of identity of the form:

$$@(P) = t$$

where "t" is a term not formed through abstraction (and the original abstraction principle settled the question when "t" is an @ -abstraction term),

[3] See e.g Wright [1983], Heck [1997], Mac Bride[2000] and Hale & Wright [2001b].

[4] Hale & Wright [2001b] and Fine [2002] are the notable exceptions. Fine's ideas inform the entire paper and have been previously discussed in Cook & Ebert [2004].

[5] Both claims have been subject to discussion within the neo-Fregean tradition. Claim (1) has been challenged by Field (see Field [1993], and Hale [1994] for a reply). Thesis (2) concerns the distinction between pure abstract objects and abstract objects more generally (see Dummett [1981] for discussion).

there remains the problem of determining whether "t" and "$@(P)$" co-refer when "t" is an abstraction term formed by the application of some abstraction operator other than "@".

The problem can be formulated as follows: Assume that we have two distinct abstraction principles:

$$\begin{aligned} AP_{@_1} &: (\forall X)(\forall Y)[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X,Y)] \\ AP_{@_2} &: (\forall X)(\forall Y)[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X,Y)] \end{aligned}$$

The question at hand is how cross-sortal identity claims of the form:

$$@_1(P) = @_2(Q)$$

(where "P " and "Q" are particular concept-expressions) are to be decided (here and below we will refer to such identities as cross-abstraction identities). Note that $AP_{@_1}$ and $AP_{@_2}$ are silent with regard to this question, and the addition of principles (1) and (2) is equally unhelpful.

We call this restricted version of the *Caesar Problem* the *C-R Problem*, since determining whether real numbers (i.e. R) might be identical to complex numbers (i.e. C) is one instance of the problem (just as determining whether Julius Caesar is a number is merely one instance of the *Caesar Problem*). While this label is both appropriate and somewhat catchy, it will turn out to be more convenient to restrict our attention, in what follows, to mathematical structures simpler than the real and complex numbers.

Importantly, even though the adoption of abstraction principles is usually associated with neo-Fregean philosophies of mathematics, one should not underestimate the generality of the *C-R Problem*, as it is not restricted to such views. Thus, although we will focus on abstractionism here, it is important to note that any philosophy of mathematics that accepts the truth of abstraction principles, even if not granting them the foundational role that neo-Fregeans attribute to them, will be confronted with the *C-R Problem*, since it owes us some story regarding how such cross-abstraction identity claims are settled within their framework. Since most platonist (or, more broadly, realist) views of mathematics, whether neo-Fregean or not, will grant the truth of such statements (or at least the truth of the corresponding Ramsey sentences), the present examination should be of general interest. Furthermore, we shall argue that appeal to equivalence classes in settling questions of cross-abstraction identity is problematic. Since this seems like the 'natural' (or at least most prevalent) means by which to solve this problem in some non-abstractionist frameworks (e.g. some variants of structuralism), we contend that the conclusions drawn here are equally relevant to platonists both Fregean and non-Fregean in character.

The remainder of this paper is partly formal and partly philosophical in character. In the next section we will briefly outline two different formal strategies (and their philosophical motivations) for dealing with the *C-R*

Problem. The first strategy will, in deciding questions of cross-abstraction identity, emphasize the role of the equivalence relations occurring on the right hand side of the abstraction principles in question. The second strategy, on the other hand, aims to settle this question in terms of the identity of the 'equivalence classes' of concepts carved out by these equivalence relations.

We will then focus on the second strategy and suggest three ways in which this strategy might be implemented to solve the *C-R Problem*. This strategy ultimately fails - each attempt to appeal to equivalence classes faces insurmountable difficulties. The first option has absurd consequences. (Interestingly we shall suggest along the way that this was Frege's own solution to the problem). The second option not only entails significant set-theoretic consequences, but is also inconsistent with what is usually taken to be a paradigm instance of an acceptable abstraction principle (namely *Hume's Principle*). Lastly, the third option is incompatible with a rather intuitive metaphysical principle - the *Principle of Uniform Identity* - which we motivate below.

2 Two Strategies

As previously mentioned, there are two broad strategies for dealing with cross-abstraction identities: Given two abstraction terms $@_1(P)$ and $@_2(Q)$ (and associated abstracts) arising from two distinct abstraction principles, we can, in deciding such identities, appeal to the identity of the corresponding equivalence relations (i.e. $E_{@_1}(P,Q)$ and $E_{@_2}(P,Q)$), or we can appeal to the identity of the associated equivalence class of concepts. Let us briefly examine the first strategy before focusing on the second.

The idea here is that, given two distinct abstraction principles:

$$\begin{aligned} AP_{@_1} &: (\forall X)(\forall Y)[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X,Y)] \\ AP_{@_2} &: (\forall X)(\forall Y)[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X,Y)] \end{aligned}$$

the cross-abstraction identity:

$$@_1(P) = @_2(Q)$$

will be true if and only if:

> "$E_{@_1}(X,Y)$" and "$E_{@_2}(X,Y)$" express the same equivalence relation.

and:

> $E_{@_1}(P,Q)$

(or, equivalently, $E_{@_2}(P,Q)$). The details of this proposal depend on how we flesh out the notion of "expresses the same equivalence relation". One straightforward means for doing so is to understand the relevant notion of sameness in terms of straightforward equivalence, so that the cross-abstraction identity is true if and only if:

$$(\forall X)(\forall Y)(E_{@_1}(X,Y) \leftrightarrow E_{@_2}(X,Y)) \wedge E_{@_1}(P,Q)$$

(equivalently $(\forall X)(\forall Y)(E_{@_1}(X,Y) \leftrightarrow E_{@_2}(X,Y)) \wedge E_{@_2}(P,Q)$).

There is an immediate problem with this approach, however. As Fine [2002] points out, this manner of settling the truth conditions of cross-abstraction identities entails that the numbers provided by *Hume's Principle*:

$$HP : (\forall X)(\forall Y)[\sharp(X) = \sharp(Y) \leftrightarrow X \approx Y]$$

(where "$X \approx Y$" abbreviates the second-order claim asserting that X and Y are equinumerous) and the numbers provided by *Finite Hume* :

$$FHP : (\forall X)(\forall Y)[\sharp(X) = \sharp(Y) \leftrightarrow (X \approx Y \vee (\neg Fin(X) \wedge \neg Fin(Y)))]$$

(where "$Fin(X)$" abbreviates the second-order formulae asserting that X is finite) are distinct, provided that the universe is uncountable. Intuitively, however, the natural numbers provided by these two principles (if both principles are acceptable) are identical - the natural numbers are a proper sub-collection of the cardinal numbers.

Thus, settling cross-abstraction identities in terms of some notion of 'sameness' of the associated equivalence relations would seem to require a more fine-grained approach to 'sameness'. However, we will neither defend the equivalence relation approach nor work out the details of such a view [6]. Instead we will outline the alternative strategy involving equivalence classes and demonstrate that no version of this approach successfully solves the *C-R Problem.*

The intuitive idea behind the second strategy is as follows: Given any abstraction operator @ and any concept P, there is a collection of concepts that 'receive' the same abstract as does P. In other words, speaking loosely, given an abstraction operator @, we can associate with each concept P the class of concepts whose @-abstracts are identical to the @-abstract of P (here $\Rightarrow$ represents an informal association):

$$@(P) \Rightarrow \{X : @(P) = @(X)\}$$

Utilizing the equivalence of "$@(Q) = @(P)$" and "$E_@(P,Q)$" this becomes:

$$@(P) \Rightarrow \{X : E_@(P,X)\}$$

[6]See Cook & Ebert [in preparation] for an examination of these issues.

The strategy that then suggests itself is the idea that cross-abstraction identities be decided in terms of the identity of the corresponding equivalence classes.

Before exploring various ways to implement the equivalence class strategy for solving the *C-R Problem*, however, we need to attend to a number of logical and metaphysical issues that will play a crucial role in what is to follow.

3 Methodology and Metaphysics

In this section we carry out a number of preliminary tasks. First, we outline a simple approach to handling abstraction principles that apply to restricted domains, i.e. subdomains of the full range of the second-order quantifiers. Next, we present two (essentially metaphysical) constraints on cross-abstraction identities that, so we argue, any successful account of cross-abstraction identity must satisfy. As we shall show, although all three variants of the equivalence class approach to the *C-R Problem* satisfy the first of these, our primary objection to the third approach (the only formally plausible one) will be its failure to satisfy the second metaphysical constraint.

3.1 Restricting Abstraction Principles

Given an abstraction principle:

$$AP_{@} : (\forall X)(\forall Y)[@(X) = @(Y) \leftrightarrow E_{@}(X, Y)]$$

we will, in what follows, often want to consider the restriction of that abstraction principle to a sub-domain of the original domain of application, say those concepts picked out by the higher-order predicate "Φ". One way of achieving this is to replace the biconditional above with a conditional whose consequent is the original abstraction principle, obtaining something like[7]:

$$AP_{@_2} : (\forall X)(\forall Y)[(\Phi(X) \wedge \Phi(Y)) \rightarrow (@_2(X) = @_2(Y) \leftrightarrow E_{@}(X, Y))]$$

This way of proceeding presents two problems. First, the principle obtained is not, as defined above, an abstraction principle at all, but rather a complex conditional embedding an abstraction principle. Second, and more worrisome, is the fact that the second-order variables here presumably still range over all concepts, and thus the abstraction operator must take a value on concepts which do not fall under the restriction Φ (something like this problem underlies worries regarding notorious neo-Fregean objects such as

[7]See Heck [1997] for a discussion of the role of such conditional versions of abstraction principles, including his variant of *Finite Hume*.

anti-zero and the "Bad" extension)[8]. We could finesse such worries in a number of ways, including adopting a free logic, or treating the referents as logical fictions, but there are more elegant ways to proceed.

Given an abstraction operator @ with its associated abstraction principle $AP_@$, we can formulate a restriction of this operator to a subdomain picked out by Φ by restricting the initial universal quantifiers to those concepts falling under Φ. In other words, the restriction of $AP_@$ to Φ will be:

$$AP_{@_2} : (\forall X_{\Phi(X)})(\forall Y_{\Phi(Y)})[@_2(X) = @_2(Y) \leftrightarrow E_@(X, Y)]$$

where:

$$(\forall X_{\Phi(X)})\Psi$$

is true if and only if all concepts satisfying Φ satisfy Ψ. Since the variables within the scope of such restricted quantifiers only range over the intended concepts, we need not worry about the unintended existence of unwanted "Bad" objects.

We can view everyday, unrestricted abstraction principles such as *Hume's Principle* as limiting cases of our notion of restricted abstraction principles, where the initial quantifiers are restricted by a predicate holding of every concept[9]. Thus, every abstraction principle, on the present approach, takes the form:

$$AP_@ : (\forall X_{\Phi(X)})(\forall Y_{\Phi(Y)})[@(X) = @(Y) \leftrightarrow E_@(X, Y)]$$

It is worth noting that our arguments won't hinge on this notational variation, rather the notion of restricted abstraction principles merely provides a simple and elegant notation for rather complicated formalisms.

With the notion of restriction in place, we can now formulate the two principles we take to be minimal constraints on a correct theory of cross-abstraction identity. It should be noted that there might be additional constrains on a solution to the *C-R Problem*, but the two given below suffice for our purposes here.

3.2 The Subsumption Constraint

As noted above, certain abstraction principles can be viewed as restrictions of other, more general, abstraction principles. One particularly useful notion regarding such restrictions is the idea that one abstraction principle subsumes another. Given two abstraction principles $AP_{@_1}$ and $AP_{@_2}$:

$$\begin{array}{lcl} AP_{@_1} & : & (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X, Y)] \\ AP_{@_2} & : & (\forall X_{\Psi(X)})(\forall Y_{\Psi(X)})[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X, Y)] \end{array}$$

[8] See Boolos [1997].

[9] In what follows we will continue to write unrestricted second-order quantifiers without the explicit restriction, i.e., "$(\forall X)\Phi$" is understood to be an abbreviation for a formula such as "$(\forall X_{(\forall z)(X(z) \leftrightarrow X(z))})\Phi$".

we will say that $AP_{@_1}$ subsumes $AP_{@_2}$ if and only if:

$$\begin{array}{rcl} (\forall X)(\Psi(X) & \rightarrow & \Phi(X)) \\ (\forall X)(\forall Y)(E_{@_1}(X,Y) & \leftrightarrow & E_{@_2}(X,Y)) \end{array}$$

In other words, $AP_{@_1}$ subsumes $AP_{@_2}$ if and only if $AP_{@_2}$ is an acceptable restriction of $AP_{@_1}$ and $E_{@_1}(X,Y)$ and $E_{@_2}(X,Y)$ agree on all pairs of concepts. Note that the second clause of the definition of subsumption is substantially stronger than the requirement that the two equivalence relations agree on the intersection of the two domains of application.

With the notion of subsumption in hand, we can place an additional constraint on any account of cross-abstraction identity. Given any two abstraction principles $AP_{@_1}$ and $AP_{@_2}$ where $AP_{@_1}$ subsumes $AP_{@_2}$, we should expect the abstracts obtained from the concepts falling under the application of both operators to be identical, i.e.:

$$(\forall X_{\Psi(X)})(@_1(X) = @_2(X))$$

The intuition behind this constraint is the same as that which motivated our earlier observation that (unrestricted) *Hume's Principle* and *Finite Hume* should generate the same finite cardinals: If $AP_{@_2}$ amounts to nothing more than restricting $AP_{@_1}$ to a subdomain of its original domain of application, then the abstracts generated on this subdomain should not change. It is worth noting that if we reformulate *Finite Hume* as an explicitly restricted version of *Hume's Principle*:

$$FHP : (\forall X_{Fin(X)})(\forall Y_{Fin(Y)})[\sharp(X) = \sharp(Y) \leftrightarrow X \approx Y]$$

then the *Subsumption Constraint* implies that abstracts generated by *Finite Hume* are a subcollection of those generated by *Hume's Principle*.[10]

3.3 The Principle of Uniform Identity

The basic idea encapsulated in the *Principle of Uniform Identity* can be motivated by the following simple and intuitive thought. Assume two abstraction principles:

$$\begin{array}{rcl} AP_{@_1} & : & (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X,Y)] \\ AP_{@_2} & : & (\forall X_{\Psi(X)})(\forall Y_{\Psi(X)})[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X,Y)] \end{array}$$

[10] One might abandon the *Subsumption Constraint* in favor of what Fine calls a "strictly separated" notion of cross-abstraction identity where each distinct pair of (acceptable) abstraction principles generate distinct ranges of objects. We do not explore this option here for two reasons. First, this approach violates the rather plausible intuitions discussed above. Second, however, implementing such an approach would require us to provide precise criteria for deciding when two abstraction principles are distinct, criteria which seem likely to cause the view to collapse into an account based on sameness of equivalence relation (see Cook and Ebert [in preparation]).

The idea is that, if it turns out that there is a concept on the shared domain of application whose $@_1$ abstract is identical to its $@_2$ abstract, then, for any concept in the shared domain, its $@_1$ abstract will be identical to its $@_2$ abstract.

As a motivational example, assume that one has two abstraction principles, the first generating extensions (including the empty extension, or set) and the other generating numbers (including, at least, zero). Further, assume that our account of cross-abstraction identity implies that zero is identical to the empty extension. Our intuition is that this is sufficient for all numbers to be identical to some sets (or, less plausible, vice versa). It would seem quite counterintutive (to say the least) to claim that some numbers introduced by, e.g. Hume's Principle, are identical to the corresponding sets while other numbers, again introduced by Hume's Principle, are not.

We can formalize this idea more precisely in the following way: Given two abstraction principles:

$$
\begin{aligned}
AP_{@_1} &: (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X,Y)] \\
AP_{@_2} &: (\forall X_{\Psi(X)})(\forall Y_{\Psi(X)})[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X,Y)]
\end{aligned}
$$

The following *Principle of Uniform Identity* should hold:

$$(\exists X_{\Phi(X)\wedge\Psi(X)})(@_1(X) = @_2(X)) \rightarrow (\forall X_{\Phi(X)\wedge\Psi(X)})(@_1(X) = @_2(X))$$

This principle seems to us rather intuitive, which might be partly due to a broadly realistic framework we both assume. We have found it difficult to provide a further theoretical defense of the principle. Nevertheless, we assume that the intuitive plausibility of the principle imposes a strong explanatory burden on anyone who wishes to explicitly deny the thesis.[11]

4 Abstracts Are Equivalence Classes

One initially promising approach to solving the *C-R Problem* is to identify the @-abstract of a concept P with the class of concepts that have the same @-abstract, i.e.:

$$@(P) = \{X : E_@(P, X)\}$$

Within an abstractionist framework, however, we need to be a bit more careful in formulation. First, we should note that sets or classes, like any other mathematical entity, will on the present view themselves be abstracts, so we will need to reformulate the above as something along the lines of:

$$@(P) = \S(Q)$$

[11] The centrality of the notion of sortal in Hale & Wright's solution to the *Caesar Problem* (Hale & Wright [2001b]) suggests that something like the *Principle of Uniform Identity* is at work in their proposed solution to the *Caesar Problem*.

where Q is itself a concept and § is the extensions (or set-forming) abstraction operator.

Intuitively, what we want is the extension of the concept holding of all concepts that are $E_{@}$-equivalent to P. In other words, loosely speaking we want Q to hold of all concepts that obtain the same @ -abstract as P. The extension operator only applies to first-level concepts, i.e. concepts that have objects as instances, and thus Q cannot literally be a concept holding of other concepts. With the extensions operator already in play, however, we can rectify this by allowing Q to be the concept holding of the extension of every concept that receives the same @-abstract as P, i.e.:

$$@(P) = \ \S((\exists Y)(z = \S(Y) \wedge E_{@}(P, Y)))$$

The use of an extension operator might already cause some doubt, since the most well known abstraction principle for providing extensions, Frege's *Basic Law V*:

$$BLV : (\forall X)(\forall Y)[\S(X) = \S(Y) \leftrightarrow (\forall z)(X(z) \leftrightarrow Y(z))]$$

is susceptible to Russell's paradox. George Boolos [1989] has proposed a means for dealing with this problem, however. The underlying idea is that some concepts are not 'well-behaved enough' to be collected into sets, or to have extensions, or to correspond to a unique abstract. Thus, we should restrict Frege's principle to certain 'well-behaved' concepts. The following *Restricted Law V* schema captures Boolos' general approach:

$$ResV : (\forall X_{\neg Bad(X)})(\forall Y_{\neg Bad(Y)})[\S(X) = \S(Y) \leftrightarrow (\forall z)(X(z) \leftrightarrow Y(z))]$$

Of course, the particular characteristics of the resulting set (or extension) theory will depend on what particular definition of "Bad" we adopt, but we need not get into such complexities here. All that we require for our purposes is that, for any abstraction operator @, each of the equivalence classes as described above will exist (and be well-behaved), i.e. for any principle $AP_{@}$ where the initial second-order quantifiers are restricted to Φ, we have:

$$(\forall X_{\Phi(X)})(\quad \neg \quad Bad(X))$$
$$(\forall X_{\Phi(X)})(\quad \neg \quad Bad((\exists Y)(x = \S(Y) \wedge E_{@}(X, Y))))$$

Simply put, the present view requires that any concept that has an abstract of any sort has an extension, and any equivalence class of concepts carved out by an acceptable equivalence relation corresponds to an extension containing exactly the extensions of concepts in that class. While greatly restricting how we might define "Bad" (and also thereby restricting what other abstraction operators we might allow into the language), these claims does not seem outright absurd.

The restriction is absurd, however, when combined with the account of cross-abstraction identity that motivated it. Fine [2002] first noticed this, writing:

> On what basis, then, could an abstract *qua* number and *qua* class be judged the same? The only reasonable view that suggests itself is that any abstract, associated through a means of abstraction with certain items, is to be identified with the class of those items. But...such a view leads to the absurd conclusion that any class C of items is identical to the class of the concept(s) whose extension is C ([2002] p. 47)

We can reconstruct Fine's reasoning as follows. The idea that any abstract is meant to be identified with the class of extensions of concepts receiving that abstract was meant to be entirely general, so, in particular, it applies to extensions themselves (without such generality we would have no means for deciding when particular extensions are or are not identical to other abstracts). Thus, for any non-"Bad" concept X:

$$\S(X) = \ \S((\exists Y)(z = \S(Y) \wedge E_{\S}(X, Y)))$$

I.e.:

$$\S(X) = \ \S((\exists Y)(z = \S(Y) \wedge (\forall w)(X(w) \leftrightarrow Y(w))))$$

This, in turn, is equivalent to:

$$\S(X) = \ \S(x = \S(X))$$

Thus, if an abstract $@(P)$ is identified with the equivalence class of concepts whose abstract is $@(P)$, then we obtain the absurd conclusion that every non-"Bad" extension is identical to its singleton.[12] This implies that the only non-"Bad" concepts, and thus the only concepts to receive abstracts of any kind, are concepts with a single instance.

Since the present proposal presupposes the acceptability of at least some abstraction operators, and presumably one of these abstraction operators will generate at least one object that is not its own singleton, we cannot identify abstracts with their corresponding equivalence classes.

5 Historical Interlude

The idea that abstracts are to be identified with their associated equivalence classes (understood in terms of extensions as above) traces back to Frege's own treatment in the *Grundlagen*, where he gives his explicit definition of numbers:

[12] The assumption that every object is identical to its singleton is not contradictory. Letting our instance of *Restricted Law V* be:

$$ResV : (\forall X_{(\exists! z)(X(z))})(\forall Y_{(\exists! z)(Y(z))})[\S(X) = \S(Y) \leftrightarrow (\forall z)(X(z) \leftrightarrow Y(z))]$$

we can construct models which satisfy the account of identity proposed above by letting the extension of each singleton property be the object that is its only instance.

> My definition is therefore as follows:
>
> The number which belongs to the concept F is the extension of the concept "equal to the concept F" (pp. 79-80)

(where, with Frege, we understand two concepts being equal in terms of there being a one-one correspondence between the two, see *Grundlagen* section 72). Thus, the number of a concept F is the extension of the concept holding of (the extension of) every concept equinumerous with F.

Frege, immediately prior to his definition of number, identifies the direction of a line a with the extension of the concept "parallel to line a", and identifies the shape of a triangle t with the extension of the concept "similar to triangle t (p. 79). It is therefore clear that Frege means the identification of abstracts with the relevant equivalence classes to be general, so that (in the second-order case) the @-abstract of a concept F is the extension holding of the concept holding of all extensions of all concepts G such that $E_{@}(F, G)$. With this observation in hand, we can make two novel observations regarding Frege's logicist reconstruction of mathematics.

First, commentators have often expressed puzzlement that Frege's definition of number as a species of extension comes at the end of a prolonged discussion of the Caesar Problem (*Grundlagen* sections 60 - 69), since this maneuver only seems to postpose the problem - we then need an account of why extensions cannot be identical to Roman emperors. We can now view Frege's intentions a bit more charitably, however. Frege's identification of abstracts with extensions can be seen as not only a reduction of the *Casear Problem* for abstracts in general to the specific case of extensions, but also as an attempted solution to what we have here called the *C-R Problem*.

As we have seen however, the proposed solution fails. Thus, there are two paradoxes lurking in Frege's *Grundlagen* and *Grundgesetze*. The first is Russell's paradox, the contradiction lurking in *Basic Law V*. Even if this problem is patched (perhaps by replacing *Basic Law V* with a Boolos-style restriction of it) there is a second contradiction, involving the incompatibility of Frege's definition of abstracts as extensions (which implies that every extension is identical to its singleton) with his claim that zero (i.e. the number of the empty concept) exists (see, e.g. *Grundlagen* p. 88). The existence of zero implies that the empty concept has a number, which, on the view of identity proposed in the last section, implies that the empty concept has an extension. The empty extension, however, cannot (on pain of contradiction) be identical to its singleton.

6 Abstracts Are Not Equivalence Classes ($ECIA_1$)

Abandoning the idea that we can solve the *C-R Problem* by identifying abstracts with their corresponding equivalence classes need not cause us to abandon the idea that the truth of such identities varies with the identity of the equivalence classes.

In order to formulate an acceptable version of this idea, we need only note that we adopted the identification of abstracts and extensions such as:

$$\begin{aligned} @_1(P) &= \S((\exists Y)(z = \S(Y) \wedge E_{@_1}(P, Y))) \\ @_2(Q) &= \S((\exists Y)(z = \S(Y) \wedge E_{@_2}(Q, Y))) \end{aligned}$$

in order to move to the following biconditional:

$$\begin{aligned} @_1(P) &= @_2(Q) \leftrightarrow \\ \S((\exists Y)(z = \S(Y) \wedge E_{@_1}(P, Y))) &= \S((\exists Y)(z = \S(Y) \wedge E_{@_1}(Q, Y))) \end{aligned}$$

Here we investigate the option of adopting this biconditional itself as our account of cross-abstraction identity, instead of deriving it from a prior, explicit account of the identity of arbitrary abstracts. Thus, given two abstraction principles:

$$\begin{aligned} AP_{@_1} &: (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X, Y)] \\ AP_{@_2} &: (\forall X_{\Psi(X)})(\forall Y_{\Psi(X)})[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X, Y)] \end{aligned}$$

cross-abstraction identities are settled via what we will call the *Equivalence Class Identity Axiom 1*:

$$\begin{aligned} ECIA_1 : (\forall X_{\Phi(X)})(\forall Y_{\Psi(Y)})[@_1(X) &= @_2(Y) \leftrightarrow \\ \S((\exists Z)(w = \S(Z) \wedge E_{@_1}(X, Z))) &= \S((\exists Z)(w = \S(Z) \wedge E_{@_1}(Y, Z)))] \end{aligned}$$

As in section 4, such an approach will require us to adopt some version of Boolos' *Restricted Law V*. As before, the approach requires that the equivalence class of (extensions of) concepts corresponding to an abstract itself be non-"Bad", i.e. for any principle $AP_@$ where the initial second-order quantifiers are restricted to Φ, we again have:

$$\begin{aligned} (\forall X_{\Phi(X)})(&\neg \; Bad(X)) \\ (\forall X_{\Phi(X)})(&\neg \; Bad((\exists Y)(x = \S(Y) \wedge E_@(X, Y)))) \end{aligned}$$

Here it is worth noting that the proposed account satisfies the *Subsumption Constraint* - given two abstraction principles where the first subsumes the second, the abstracts generated on their shared domain of application will be identical. Unlike the previous account, no outright absurd consequences follow from $ECIA_1$

We do, however, obtain some rather surprising results. Consider the claim that equivalence classes of (extensions of) concepts must be non-"Bad" if the associated abstracts are, and apply it to the extension operator itself, obtaining:

$$(\forall X_{\neg Bad(X)})(\neg Bad((\exists Y)(x = \S(Y) \wedge E\S(X, Y))))$$

which is just:

$$(\forall X_{\neg Bad(X)})(\neg Bad((\exists Y)(x = \S(Y) \wedge (\forall w)(X(w) \leftrightarrow Y(w)))))$$

i.e.:

$$(\forall X_{\neg Bad(X)})(\neg Bad(x = \S(X)))$$

This, however, states that the concept holding solely of the extension of a non-"Bad" concept is itself non-"Bad", i.e. the singleton of any extension is also an extension. Thus, this account of cross-abstraction identity gives us arbitrary singletons of extensions for free.

Actually, we get quite a bit more than this, as the following example will illustrate. Assume that (in addition to an appropriate version of *Restricted Law V*) unrestricted *Hume's Principle* is an acceptable abstraction principle:

$$HP : (\forall X)(\forall Y)[\sharp(X) = \sharp(Y) \leftrightarrow X \approx Y]$$

Now, since

$$(\forall X_{\Phi(X)})(\neg Bad(X))$$

(and in the case of unrestricted *Hume's Principle*, $\Phi(X)$ is$(\forall y)(X(y) \leftrightarrow X(y)))$ it follows that:

$$(\forall X)\neg Bad(X)$$

Of course, *Restricted Law V* is restricted to the non-"Bad" concepts, so, if the non-"Bad" concepts are just all concepts, then we have an instance of *Basic Law V*. In other words, on this view unrestricted *Hume's Principle* is inconsistent, contrary to the widespread view that *Hume's Principle* is a paradigm instance of an acceptable abstraction principle, if anything is.

Further assume that we have some restricted version of *Hume's Principle*:

$$HP : (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[\sharp(X) = \sharp(Y) \leftrightarrow X \approx Y]$$

where the empty property and all properties with only a single instance fall in the domain of application:

$$(\forall X)((\forall y)(\forall z)((X(y) \wedge X(z)) \rightarrow y = z) \rightarrow \Phi(X))$$

Now, since

$$(\forall X_{\Phi(X)})(\neg Bad(X))$$

it follows that:

$$\neg Bad(x \neq x)$$

In other words, the empty set exists. Additionally, since any concept with exactly one instance has a number, it follows that all single instance concepts have extensions. Combining this with:

$$(\forall X_{\Phi(X)})(\neg Bad((\exists Y)(x = \S(Y) \wedge X \approx Y)))$$

provides an extension holding of all single membered extensions, i.e. the extension (or set) of all singletons.[13]

Thus, combining $ECIA_1$ with a very weak version of the number abstraction principle implies some surprisingly strong theorems regarding extensions. In particular, we get that the empty extension (set) exists, that the singleton of any abstract (and thus of any extension) exists, and that the (infinite) extension of all singletons exists. This (of course) via Russell's paradox style reasoning, rules out the truth of the separation axiom for extensions.

Thus, although there is nothing inconsistent about this approach, it does entail substantial set-theoretic principles. Such a result is surprising, since we have (presumably) said nothing significant at this point about extensions, and merely adopted a simple account of cross-abstraction identity. Such a situation might make us wary of any attempt to settle cross-abstraction identities in terms of the identities of the corresponding equivalence classes, understood in this context to be extensions (thereby privileging one sort of abstract over the rest).

7 No Need For Equivalence Classes ($ECIA_2$)

In the previous sections we have explored ways of settling cross-abstraction identities in terms of the identity of the corresponding equivalence classes, where we took the existence of such classes seriously, defining them to be certain extensions (and thus assuming that such extensions existed in order to do the work required of them). Abandoning explicit use of extensions, however, need not force us to abandon the more general idea that cross-abstraction identities are to be settled in terms of whether or not the abstracts in question correspond to the same 'collection' of concepts. The question facing us at this point is how to formulate such an idea without the use of extensions.

The answer is that we can paraphrase away all reference to the equivalence classes. The initial idea was that a cross-abstraction identity:

$$@_1(P) = @_2(Q)$$

is true if and only if the corresponding equivalence relations are identical:

$$\{X : E_{@_1}(P, X)\} = \{X : E_{@_2}(Q, X)\}$$

Ignoring for the moment what sort of entity such equivalence classes might be, the above is, by (a second order analogue of) the axiom of extensionality,

[13]Note that this result can be adapted to show that, on the present account of identity, any instance of *Restricted Law V* (such as *NewV*) which implies that there are no non-"Bad" concepts equinumerous with the universe is inconsistent with any restricted version of *Hume's Principle* that implies the existence of any number other than 0.

equivalent to:

$$(\forall Y)(Y \in \{X : E_{@_1}(P, X)\} \leftrightarrow Y \in \{X : E_{@_2}(Q, X)\})$$

which, intuitively, is just:

$$(\forall Y)(E_{@_1}(P, Y) \leftrightarrow E_{@_2}(Q, Y))$$

Thus, we can eliminate all explicit talk of equivalence classes themselves in favor of the above biconditional.[14] Given any two abstraction principles:

$$\begin{aligned} AP_{@_1} &: (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[@_1(X) = @_1(Y) \leftrightarrow E_{@_1}(X, Y)] \\ AP_{@_2} &: (\forall X_{\Psi(X)})(\forall Y_{\Psi(X)})[@_2(X) = @_2(Y) \leftrightarrow E_{@_2}(X, Y)] \end{aligned}$$

cross-abstraction identities are settled by the *Equivalence Class Identity Axiom 2*:[15]

$$ECIA_2 : (\forall X_{\Phi(X)})(\forall Y_{\Psi(X)})[@_1(X) = @_2(Y) \leftrightarrow (\forall Z)(E_{@_1}(X, Z) \leftrightarrow E_{@_2}(Y, Z))$$

(Note that the internal second-order universal quantifier "$(\forall Z)$" is unrestricted.)

We can get a feel for the content of the $ECIA_2$ by examining a particular instance. Consider *Hume's Principle* and the instance of the *Restricted Law V* schema known as *NewV*:

$$NewV : (\forall X_{\neg Big(X)})(\forall Y_{\neg Big(Y)})[\S(X) = \S(Y) \leftrightarrow (\forall z)(X(z) \leftrightarrow Y(z))]$$

(Where "Big(X)" is an abbreveation for the second-order formula asserting that X is equinumerous to the entire domain.) Given the relevant instance of the *Equivalence Class Identity Axiom 2*:

$$(\forall X)(\forall Y_{\neg Big(Y)})[\sharp(X) = \ \S(Y) \leftrightarrow (\forall Z)(E_{\sharp}(X, Z) \leftrightarrow E_{\S}(X, Z))]$$

i.e.:

$$\begin{aligned} (\forall X)(\forall Y_{\neg Big(Y)})[\sharp(X) = \ \S(Y) &\leftrightarrow (\forall Z)(X \approx Z \leftrightarrow ((\forall z)(X(z) \\ &\leftrightarrow Y(z)) \vee (Big(X) \wedge Big(Y)))))] \end{aligned}$$

we can prove:

$$\sharp(X) = \S(Y) \leftrightarrow (\forall z)(\neg X(z) \wedge \neg Y(z))$$

[14] One might have epistemological worries regarding an account motivated in terms of entities, i.e. equivalence classes, whose existence is then 'paraphrased' away. We will not dwell on this issue however, since as we shall see, there are deeper problems with this strategy.

[15] This principle is adopted by Fine [2002], although its incorporation into his account owes more to its mathematical elegance than any deep philosophical commitment.

In other words (assuming for the sake of argument that these are both acceptable abstraction principles), the number 0 is identical to the empty set $\emptyset$, and no other number is identical to any other extension.[16] This result should strike one as rather surprising.

Recall that we required that any account of cross-identity abstraction should be consistent with the *Principle of Uniform Identity*, which asserts that, given two abstraction principles $AP_{@_1}$ and $AP_{@_2}$ (restricted to $\Phi(X)$ and $\Psi(X)$ respectively):

$$(\exists X_{\Phi(X) \wedge \Psi(X)})(@_1(X) = @_2(X)) \rightarrow (\forall X_{\Phi(X) \wedge \Psi(X)})(@_1(X) = @_2(X))$$

$ECIA_2$ is consistent with this claim, since we can suppose that some version of *Restricted Law V* is the only acceptable abstraction principle (thus rendering the *Principle of Uniform Identity*, and the issue of cross-abstraction identity, irrelevant). The problem lies, however, in the fact that $ECIA_2$ plus the *Principle of Uniform Identity* renders the simultaneous existence of both zero and the empty set extremely problematic.

Assume that both zero and the empty set exist, i.e. some instance of *Restricted Law V*:

$$ResV : (\forall X_{\neg Bad(X)})(\forall Y_{\neg Bad(Y)})[\S(X) = \S(Y) \leftrightarrow (\forall z)(X(z) \leftrightarrow Y(z))]$$

where the empty concept is not-"Bad", and any restricted version of *Hume's Principle*:

$$HP : (\forall X_{\Phi(X)})(\forall Y_{\Phi(X)})[\sharp(X) = \sharp(Y) \leftrightarrow X \approx Y]$$

where the empty concept falls under Φ. Given these principles, $ECIA_2$ tells us that the empty set ($\S(\neg x = x)$) and zero ($\sharp(\neg x = x)$) both exist and are identical. As a result, if we assume the *Principle of Uniform Identity*, it follows that no concept (other than the empty one) can both be non-"Bad" and fall under Φ (since non-empty concepts will correspond to different equivalence classes relative to *Restricted Law V* and *Hume's Principle* and thus no concept (other than the empty one) can have both a number and an extension. As a result, the abstractionist who adopts $ECIA_2$ is faced with a rather uncomfortable trilemma:

[1] One of zero and the empty set fails to exist.

[2] No concept, other than the empty one, has both a number and an extension (i.e. set).

[3] The *Principle of Uniform Identity* fails.

[16] Note that, on the present way of formulating things, we need not worry about the possible identity of the "Bad" extension and anti-zero, since only the latter exists.

Presumably, for an abstractionist looking to provide a foundation for mathematics based on the existence of abstract objects, access to which is given by abstraction principles, accepting [2] is a non-starter, and [1] would seem to fare little better. Thus, $ECIA_2$ seems to imply the failure of the *Principle of Uniform Identity.*

8 Conclusion

As we have seen, all three ways of pursuing the idea that cross-abstraction identities can be settled in terms of the identity of the corresponding class of concepts fail. Identifying abstracts with the corresponding equivalence classes leads to absurdities; allowing the identities to co-vary with the identities of the corresponding classes understood as extensions turned out both to be incompatible with unrestricted *Hume's Principle* and to imply significant set-theoretic principles; and paraphrasing away the existence of these classes in terms of second-order quantification left us with a view that is incompatible with the *Principle of Uniform Identity.* We see no other way of implementing the idea that cross-abstraction identities be settled in terms of sameness of the corresponding 'collection' of concepts.

As a result the proponent of such an approach would seem to have no other option than to accept $ECIA_2$ and find some independent reason for rejecting the *Principle of Uniform Identity.* However, for the considerations sketched earlier, we are doubtful that principled reasons can be given for abandoning this basic metaphysical principle.

Thus, the only other option for solving the *C-R Problem* is to invoke sameness of equivalence relations (as sketched in section 2). Successfully carrying out a defence of this idea will require, among other things, a formal account of the notion of identity between equivalence relation. We plan on addressing this problem in a sequel.[17]

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